

LIFTING HARMONIC MORPHISMS OF TROPICAL CURVES, METRIZED COMPLEXES, AND BERKOVICH SKELETA

OMID AMINI, MATTHEW BAKER, ERWAN BRUGALLÉ, AND JOSEPH RABINOFF

ABSTRACT. Let K be a complete and algebraically closed field with value group Λ and residue field k , and let $\varphi : X' \rightarrow X$ be a finite morphism of smooth, proper, irreducible, stable marked algebraic curves over K . We show that φ gives rise in a canonical way to a finite and effective harmonic morphism of Λ -metric graphs, and more generally to a finite harmonic morphism of Λ -metrized complexes of k -curves. These canonical “abstract tropicalizations” are constructed using Berkovich’s notion of the skeleton of an analytic curve. Our arguments give analytic proofs of stronger “skeletonized” versions of some foundational results of Liu-Lorenzini, Coleman, and Liu on simultaneous semistable reduction of curves.

We then consider the inverse problem of lifting finite harmonic morphisms of metric graphs/tropical curves and metrized complexes to morphisms of curves over K . We prove that every tamely ramified finite harmonic morphism of Λ -metrized complexes of k -curves lifts to a finite morphism of K -curves. If in addition the ramification points are marked, we obtain a complete classification of all such lifts along with their automorphisms. This generalizes and provides new analytic proofs of earlier results of Saïdi and Wewers. We prove a similar result concerning the existence of liftings for morphisms of tropical curves, except the genus of the source curve can no longer be fixed. From this point of view, morphisms of metrized complexes are better behaved than morphisms of tropical curves. The caveat on the genus in the lifting result for tropical curves is necessary: we show by example that the gonality of a tropical curve C can be strictly smaller than the gonality of any smooth proper curve X of the same genus lifting C .

We also give various applications of these results. For example, we show that linear equivalence of divisors on a tropical curve C coincides with the equivalence relation generated by declaring that the fibers of every finite harmonic morphism from C to the tropical projective line are equivalent. We study liftability of metrized complexes equipped with a finite group action, and as an application classify all (augmented) metric graphs arising as the tropicalization of a hyperelliptic curve. We also discuss the relationship between harmonic morphisms of metric graphs and induced maps between component groups of Néron models, providing a negative answer to a question of Ribet motivated by number theory.

Throughout this paper, unless explicitly stated otherwise, K denotes an algebraically closed field which is complete with respect to a nontrivial non-Archimedean valuation $\text{val} : K \rightarrow \mathbf{R} \cup \{\infty\}$. Its valuation ring is denoted R , its maximal ideal is \mathfrak{m}_R , and the residue field is $k = R/\mathfrak{m}_R$. We denote the value group of K by $\Lambda = \text{val}(K^\times) \subset \mathbf{R}$.

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1. INTRODUCTION

1.1. General overview. The basic motivation behind the investigations in this paper is to understand the relationship between tropical and algebraic curves. A fundamental question along these lines is the following:

(Q) Which morphisms between tropical curves arise as tropicalizations¹ of morphisms of algebraic curves?

In addition to the lifting problem for morphisms of tropical curves, we also study questions such as “Which tropical curves arise as tropicalizations of hyperelliptic curves?”. This naturally leads us to study group actions on tropical curves and how notions such as gonality change under tropicalization. We also investigate applications of the lifting problem to arithmetic geometry; for example, we use our results to provide a negative answer to a question of Ribet about induced maps between component groups of Néron models.

In this paper we will consider three different kinds of “tropical” objects which one can associate to a smooth, proper, connected algebraic curve X/K , each depending on the choice of a *triangulation* of X . Roughly speaking, a triangulation $(X, V \cup D)$ of X (with respect to a finite set of punctures $D \subset X(K)$) is a finite set V of points in the Berkovich analytification X^{an} of X whose removal partitions X^{an} into open balls and finitely many open annuli (with the punctures belonging to distinct open balls). Triangulations of (X, D) are naturally in one-to-one correspondence with semistable models \mathfrak{X} of (X, D) ; see Section 5. The skeleton of a triangulated curve is the dual graph of the special fiber \mathfrak{X}_k of the corresponding semistable model, equipped with its canonical metric.

To any triangulated curve, one may associate the three following “tropical” objects, at each step adding some additional structure:

- (1) a *metric graph* Γ : this is the skeleton of the triangulated curve $(X, V \cup D)$;
- (2) an *augmented metric graph* (Γ, g) , i.e., a metric graph Γ enhanced with a genus function $g : \Gamma \rightarrow \mathbf{Z}_{\geq 0}$ which is non-zero only at finitely many points: this is the above metric graph together with the function g satisfying $g(p) = 0$ for $p \notin V$ and $g(v) = \text{genus}(C_v)$ for $v \in V$, where C_v is the (normalization of the) irreducible component of \mathfrak{X}_k corresponding to v ;
- (3) a *metrized complex of curves* \mathcal{C} , i.e., an augmented metric graph Γ equipped with a vertex set V and a punctured algebraic curve over k of genus $g(v)$ for each point $v \in V$, with the punctures in bijection with the tangent directions to v in Γ : this is the above metric graph, together with the curves C_v for $v \in V$ and punctures given by the singular points of \mathfrak{X}_k .

An (augmented) metric graph or metrized complex of curves arising from a triangulated curve by the above procedure is said to be *liftable*. If $(X, V \cup D)$ and $(X, V' \cup D')$ are triangulations of the same curve X , with $D' \subset D$ and $V' \subset V$, then the corresponding metric graphs are related by a so-called *tropical modification*. Tropical modifications generate an equivalence relation on the set of (augmented) metric graphs, and an equivalence class for this relation is called an *(augmented) tropical curve*. The (augmented) *tropicalization* of a K -curve X is by definition the (augmented) tropical curve C corresponding to any triangulation of X . Tropical curves and augmented tropical curves can be thought of as “purely combinatorial” objects, whereas metrized complexes are a mixture of combinatorial objects (which one thinks of as living over the *value group* Λ of K) and algebro-geometric objects over the residue field k of K .

There is a natural notion of *finite harmonic morphism* between metric graphs which induces a natural notion of *tropical morphism* between tropical curves. There is a corresponding notion of tropical morphism for *augmented* tropical curves, where in addition to the harmonicity condition one

¹In the present paper tropicalization is defined via Berkovich’s theory of analytic spaces (see also [Pay09], [BPR11], [CLD12]). Another framework for tropicalization has been proposed by Kontsevich-Soibelman [KS01] and Mikhalkin (see for example [Mik06]), where the link between tropical geometry and complex algebraic geometry is provided by real one-parameter families of complex varieties. For some conjectural relations between the two approaches see [KS01, KS06].

imposes a “Riemann-Hurwitz condition” that the ramification divisor is effective. There is also a natural notion of finite harmonic morphism for metrized complexes of curves. Each kind of object (metric graphs, tropical curves, augmented tropical curves, metrized complexes) forms a category with respect to the corresponding notion of morphism. The construction of an (augmented) tropical curve C (resp. metrized complex \mathcal{C}) out of a triangulated K -curve X is functorial, in the sense that a finite morphism of triangulated K -curves $X' \rightarrow X$ induces in a natural way a tropical morphism $C' \rightarrow C$ (resp. a finite harmonic morphism $\mathcal{C}' \rightarrow \mathcal{C}$).

Our original question (Q) now breaks up into the following two separate questions:

- (Q1) Which finite harmonic morphisms of metrized complexes can be lifted to finite morphisms of triangulated curves?
- (Q2) Which tropical morphisms between augmented tropical curves can be lifted to finite harmonic morphisms of metrized complexes?

One can also forget the augmentation function $g : \Gamma \rightarrow \mathbf{Z}_{\geq 0}$ and ask the following variant of (Q2):

- (Q2') Which tropical morphisms between tropical curves can be lifted to finite harmonic morphisms of metrized complexes?

One of our main results is that the answer to question (Q1) is essentially “all”, so the situation here is rather satisfactory; there is no obstruction to lifting, at least assuming everywhere tame ramification when k has characteristic $p > 0$. With some additional natural assumptions, we actually prove a much stronger statement, providing a complete classification of the set of isomorphism classes of liftings. We stress that by definition, the genus and degree are automatically preserved by such lifts.

Essentially by definition, (Q2) reduces to an existence problem for ramified coverings $\varphi_{v'} : C'_{v'} \rightarrow C_v$ of a given degree with some prescribed ramification profiles. Hence the answer to (Q2) is intimately linked with the question of non-vanishing of Hurwitz numbers. In particular one can easily construct tropical morphisms between augmented tropical curves which cannot be promoted to a finite harmonic morphism of metrized complexes (and hence cannot be lifted to a finite morphism of smooth proper curves over K). The simplest example of such a tropical morphism is depicted in Figure 1, and corresponds to the classical fact that although it would not violate the Riemann-Hurwitz formula, there is no degree 4 map of smooth proper connected curves over \mathbf{C} having ramification profile $\{(2, 2), (2, 2), (3, 1)\}$; this is a consequence of the (easy part of the) Riemann Existence Theorem (see Example 8.4 below for more details).

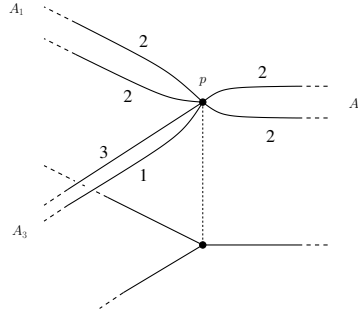


FIGURE 1. A tropical morphism of degree four which cannot be promoted to a degree 4 morphism of metrized complexes of curves. The labels on the edges are the “expansion factors” of the harmonic morphism, see Definition 2.4 below.

Understanding when Hurwitz numbers vanish remains mysterious in general, so at present there is no satisfying “combinatorial” answer to question (Q2), in which we require that the genus of the objects in question be preserved by our lifts. However, if we drop the latter condition, i.e., if we consider instead question (Q2’), we will see that the answer to (Q2’) is also “all”.

We prove a number of additional results which supplement and provide applications of the above results. For example:

- We discuss the relationship between harmonic morphisms of metric graphs and component groups of Néron models, connecting our lifting results to natural questions in arithmetic algebraic geometry. As a concrete application, we provide a negative answer to the following question of Ribet: Suppose $f : X' \rightarrow X$ is a finite morphism of semistable curves over a discretely valued field K_0 with $g(X) \geq 2$. If the special fiber of the minimal regular model of X' consists of two projective lines intersecting transversely, is the induced map $f_* : \Phi_{X'} \rightarrow \Phi_X$ on component groups of Néron models necessarily surjective?
- When the target curve has genus zero, we investigate a variant of question (Q2') in which the genus of the source curve may be prescribed, at the cost of losing control over the degree of the morphism. As an application, we show that linear equivalence of divisors on a tropical curve C coincides with the equivalence relation generated by declaring that the fibers of every tropical morphism from C to the tropical projective line \mathbf{TP}^1 (see Example 2.31) are equivalent.
- We study the lifting problem for a metrized complex \mathcal{C} equipped with a *tame action* of a finite group H . We prove that a lifting exists if and only if the quotient \mathcal{C}/H exists in the category of metrized complexes, and we discuss the existence of such quotients. As an application, we study hyperelliptic augmented metric graphs in detail, relating them to harmonic morphisms and characterizing liftability to hyperelliptic curves in a combinatorial way.
- We provide counterexamples to certain plausible-sounding statements about the connection between ranks of divisors on tropical curves and harmonic morphisms. For example, it is not true that a degree d and rank 1 divisor on a tropical curve Γ for which $|D|$ is base-point free appears as the fiber of some finite harmonic morphism from Γ to a tropical curve of genus 0. We also show by example that the *gonality* of an augmented tropical curve C (i.e., the minimal degree of a tropical morphism from C to \mathbf{TP}^1) can be strictly smaller than the gonality of any smooth proper curve X lifting C .

Organization of the paper. The paper is organized as follows. We spend the rest of this section presenting a more detailed overview of the main definitions and results of this paper. Precise definitions of metric graphs, tropical curves, and metrized complexes of curves, as well as various kinds of morphisms between these objects, are given in Section 2. In that section we also provide a number of examples. Sections 3 and 4 provide a detailed account of tropicalization via Berkovich's theory of analytic curves. The metrized complex of k -curves associated to a triangulated punctured K -curve is described in Section 3, and tropicalizations of morphisms of triangulated curves are treated in Section 4, which includes a rigorous development of various results concerning the maps between skeleta induced by a finite morphism of Berkovich analytic curves. This is closely related to the “simultaneous semistable reduction” theorem of Liu-Lorenzini [LL99], and also to the notion of the “stable hull of a morphism” due to Coleman [Col03] and Liu [Liu06]. We discuss these relations in Section 5. We prove our lifting results for metrized complexes in Sections 6 and 7. Lifting results for (augmented) metric graphs and tropical curves are proved in Section 8. Section 9 contains applications of our lifting results. For example, the relation with component groups of Néron models is discussed in (9.1) and lifting results for metrized complexes equipped with a finite group action are discussed in (9.11). In (9.11) we also give a complete classification of all hyperelliptic augmented tropical curves which can be realized as the minimal skeleton of a hyperelliptic curve. Finally, in Section 10 we study tropical rank and gonality and related lifting questions.

1.2. Metric graphs and metrized complexes. We now make the above discussion more precise, presenting some rigorous definitions and formulating in a precise way some of our main results.

Λ -metric graphs and models. Let Λ be a subgroup of the real numbers. A Λ -metric graph is the geometric realization of a finite edge-weighted graph G with edge weights belonging to $\Lambda \cup \{\infty\}$ (where only 1-valent vertices of G are allowed to be at infinite distance). We call G a *model* for Γ . By fixing a model we can talk about vertices and edges of Γ .

Augmented metric graphs. An *augmented Λ -metric graph* is a Λ -metric graph Γ along with a function $g : \Gamma \rightarrow \mathbf{Z}_{\geq 0}$, called the *genus function*, such that $g(p) = 0$ for all infinite vertices and $g(p) \neq 0$ for only finitely many points $p \in \Gamma$. The *genus* of Γ is defined to be $g(\Gamma) = h_1(\Gamma) + \sum_{p \in \Gamma} g(p)$. There is also a *canonical divisor* on Γ defined by $K_\Gamma = \sum_{p \in \Gamma} (\text{val}(p) + 2g(p) - 2)(p)$.

Morphisms of metric graphs. Let Γ' and Γ be Λ -metric graphs and let $\varphi : \Gamma' \rightarrow \Gamma$ be a continuous map. The map φ is called a *morphism of Λ -metric graphs* if there are models G' and G for Γ' and Γ , respectively, such that $\varphi(V(\Gamma')) \subset V(\Gamma)$, $\varphi^{-1}(E(\Gamma)) \subset E(\Gamma')$, and the restriction of φ to any edge e' of Γ' is a dilation by some factor $d_{e'}(\varphi) \in \mathbf{Z}_{\geq 0}$. We call the morphism φ *finite* if $d_{e'}(\varphi) > 0$ for any edge $e' \in E(\Gamma')$. This latter holds if and only if φ has finite fibers.

Harmonic morphisms of metric graphs. Fix $v' \in V(\Gamma')$ and let $v = \varphi(v')$. The morphism φ is said to be *harmonic at v'* provided that, for every edge e adjacent to v , the number $d_{v'}(\varphi) := \sum_{e' \mapsto e} d_{e'}(\varphi)$ is independent of e , where the sum is taken over all edges e' adjacent to v' which map onto e . (This definition must be modified slightly when Γ has loop edges; see Section 2.) We say that φ is *harmonic* if it is *surjective* and harmonic at all $v' \in V(\Gamma')$; in this case the number $\deg(\varphi) = \sum_{v' \mapsto v} d_{v'}(\varphi)$ is independent of $v \in V(\Gamma)$, and is called the *degree* of φ (see Example 2.8).

Ramification. Let $\varphi : \Gamma' \rightarrow \Gamma$ be a harmonic morphism of augmented Λ -metric graphs and define the *ramification divisor* $R = \sum_{v'} R_{v'}(v')$ for φ by the *Riemann-Hurwitz formula* $R = K_{\Gamma'} - \varphi^*(K_\Gamma)$. We say that a finite harmonic morphism $\varphi : \Gamma' \rightarrow \Gamma$ of augmented Λ -metric graphs is *effective* if R is effective outside the infinite vertices of Γ' . We say that φ is *étale* if $R = 0$ and *generically étale* if R is supported on the set of infinite vertices of Γ' (see Example 2.18).

Metrized complexes of curves. Let k be an algebraically closed field and let Λ be as above. A Λ -metrized complex of k -curves consists of the following data (see Example 2.21):

- An augmented Λ -metric graph Γ equipped with the choice of a distinguished model G .
- For every finite vertex v of Γ , a smooth, proper, connected k -curve C_v of genus $g(v)$.
- An injective function red_v , called the *reduction map*, from the set of tangent directions at v in Γ to $C_v(k)$. We call the image of red_v the set of *marked points* on C_v .

Harmonic morphisms of metrized complexes of curves. A *harmonic morphism* of metrized complexes of curves consists of a harmonic morphism $\varphi : \Gamma' \rightarrow \Gamma$ of augmented metric graphs (respecting the given models), and for every finite vertex v' of Γ' with $d_{v'}(\varphi) > 0$ a finite morphism of curves $\varphi_{v'} : C'_{v'} \rightarrow C_{\varphi(v')}$, satisfying the natural compatibility conditions listed in Definition 2.22.

A harmonic morphism of metrized complexes of curves is called *finite* if the underlying harmonic morphism of metric graphs is finite. Metrized complexes of curves with harmonic morphisms between them form a category, with finite harmonic morphisms giving rise to a subcategory.

Tame coverings of metrized complexes of curves. Let $\varphi : \mathcal{C}' \rightarrow \mathcal{C}$ be a finite harmonic morphism of metrized complexes of curves. If $\text{char}(k) = 0$, we say that φ is a *tame covering* if $\varphi : \Gamma' \rightarrow \Gamma$ is a generically étale finite morphism of augmented metric graphs. If $\text{char}(k) = p > 0$, we require in addition that $d_{e'}(\varphi)$ is prime to p for all edges e' of Γ' , and that $\varphi_{v'}$ is separable for all finite vertices v' of Γ' .

1.3. Triangulated curves and finite morphisms. A significant portion of the paper is devoted to a rigorous development of various results concerning the maps between skeleta induced by a finite morphism of Berkovich analytic curves. This is closely related to the “simultaneous semistable reduction” theorem of Liu-Lorenzini [LL99], and also to the notion of the “stable hull of a morphism” due to Coleman [Col03] and Liu [Liu06]. In addition to being crucial for the logical development of

our paper, this material should also be useful for other purposes so we have proved certain results in greater generality than is logically necessary for the present paper alone. We assume in this section that the reader is familiar with the basics of the theory of Berkovich analytic curves; for background see for example [Tem] and [BPR11, §5].

Analytic curves. For the purposes of this introduction, by an *analytic curve* we will mean the Berkovich analytification X of a smooth, connected, projective algebraic K -curve. As in [BPR11, 5.15], we define a *semistable vertex set* of X to be a finite set V of type-2 points of X such that $X \setminus V$ is a disjoint union of open balls and finitely many open annuli. More generally, we consider analytic curves X together with a finite set $D \subset X(K)$ of marked points (which we call *punctures*).

Semistable vertex sets and triangulations. A *semistable vertex set* of (X, D) is a semistable vertex set V of X such that the points of D are contained in distinct open ball connected components of $X \setminus V$. A *triangulated punctured curve* $(X, V \cup D)$ is a triple consisting of an analytic curve X as above together with a finite set D of punctures and a semistable vertex set V for (X, D) . If $(X, V \cup D)$ is a triangulated punctured curve, the complement $X \setminus (V \cup D)$ is a disjoint union of open balls and finitely many generalized open annuli (open annuli or punctured open balls). We call this the associated *semistable decomposition*.

A triangulation is the non-Archimedean analogue of a pair-of-pants decomposition of a Riemann surface: it gives a way of decomposing the curve X into pieces which are attached along common open annuli.

Skeleton of a triangulated punctured curve. There is a Λ -metrized complex of k -curves $\Sigma(X, V \cup D)$ canonically associated to a triangulated punctured K -curve $(X, V \cup D)$. The Λ -metric graph $\Gamma(X, V \cup D)$ underlying $\Sigma(X, V \cup D)$ is the skeleton of $X \setminus D$ relative to V , as defined for example in [BPR11, 5.17]: the finite vertices of $\Gamma(X, V \cup D)$ are the points of V , the infinite vertices are the points of D , and the edges of $\Gamma(X, V \cup D)$ are the skeleta of the generalized open annuli in the associated semistable decomposition. The k -curve C_v associated to a vertex $v \in V$ is the unique smooth projective connected curve over k whose function field is the double residue field $\widetilde{\mathcal{H}}(v)$ of the point $v \in X$. The marked points of C_v correspond in a natural way to tangent directions at v in $\Gamma(X, V \cup D)$.

We prove in Theorem 3.24 that every Λ -metrized complex of k -curves arises from this construction.

Minimal triangulations. If the Euler characteristic $\chi(X, D) := 2 - 2g(X) - \#D$ is negative (in which case we say that the punctured curve $X \setminus D$ is *hyperbolic*), then there is a canonical *minimal* semistable vertex set V for (X, D) , i.e., (X, D) has a canonical triangulation, which we call the *stable triangulation* of (X, D) . In particular, if $g(X) \geq 2$, $g(X) = 1$ and $\#D \geq 1$, or $g(X) = 0$ and $\#D \geq 3$, then there is a canonical way to associate a metrized complex to the punctured curve $X \setminus D$.

Semistable vertex sets and semistable models. If $D = \emptyset$, there are canonical bijections between semistable vertex sets for X , semistable admissible formal R -models for X , and semistable algebraic R -models for X . In general, there is a canonical bijection between semistable vertex sets for (X, D) and semistable admissible formal R -models (resp. semistable algebraic R -models) for X in which the points of D reduce to smooth points on distinct irreducible components of the special fiber (see Theorem 5.8). So one can translate back and forth between the language of semistable vertex sets and semistable models.

Finite morphisms of triangulated curves and metrized complexes. We now discuss morphisms between triangulated curves and their relationship to morphisms of metrized complexes. A *finite morphism* $\varphi : (X', V' \cup D') \rightarrow (X, V \cup D)$ of triangulated punctured curves is a finite morphism $\varphi : X' \rightarrow X$ such that $\varphi^{-1}(D) = D'$, $\varphi^{-1}(V) = V'$, and $\varphi^{-1}(\Sigma(X, V \cup D)) = \Sigma(X', V' \cup D')$ as sets. We prove in Corollary 4.28 that a finite morphism of triangulated punctured K -curves naturally induces a finite harmonic morphism of Λ -metrized complexes of k -curves. We also prove, in Corollary 4.26 (see also Theorems 5.25 and 5.13) that every finite morphism of (punctured) K -curves can be extended to a finite morphism of *triangulated* (punctured) curves. This can be seen as a skeleton-theoretic

simultaneous semistable reduction theorem relative to a finite morphism of curves. In fact, together with a simple descent argument, one can formally derive from these results a theorem of Liu [Liu06] which says that if $\varphi : X' \rightarrow X$ is a finite morphism of smooth, proper, geometrically connected K_0 -curves, where K_0 is a discretely valued field, then there is a finite extension K'_0 of K_0 such that φ extends to a finite morphism of semistable models over the valuation ring of K'_0 .

Stable hull of a finite morphism of curves. Under certain supplementary hypotheses, there exists a canonical *minimal* finite morphism of triangulated punctured curves extending a given finite morphism $\varphi : X' \rightarrow X$ of K -curves. (Beware that the minimal skeleta with respect to the morphism φ may be larger than the minimal skeleta of the curves X, X' .) As above, it follows formally that if $\varphi : X' \rightarrow X$ is a finite morphism of smooth, proper, geometrically connected K_0 -curves, then there exist minimal semistable models of X and X' (defined over a finite, separable extension of K_0) such that φ extends to a finite morphism of models. Liu [Liu06] calls this minimal extension the *stable hull* of the morphism φ . In particular, we obtain a new analytic proof of Liu's original theorem (a special case of which had been proved earlier by Coleman [Col03]) as well as a generalization to non-discretely valued fields. We emphasize that it is the skeletal version of the theorem that we need: it implies the existence of minimal semistable models \mathcal{X}' and \mathcal{X} of X' and X , defined over a finite separable extension of K_0 , such that φ extends to a finite morphism $\varphi : \mathcal{X}' \rightarrow \mathcal{X}$ with the extra property that $\varphi^{-1}(\mathcal{X}_k^{\text{sing}}) = \mathcal{X}'_k{}^{\text{sing}}$ over the special fibers; our theorem does not follow directly from Liu's.²

Tame coverings of triangulated punctured curves. Let $(X', V' \cup D')$ be a triangulated punctured curve with skeleton $\Sigma' = \Sigma(X', V' \cup D')$ and let $\varphi : (X', V' \cup D') \rightarrow (X, V \cup D)$ be a finite morphism. If $\text{char}(k) = 0$, we say that φ is a *tame covering* of $(X, V \cup D)$ provided that D' and D contain the branch points of φ and their images, respectively. If $\text{char}(k) = p > 0$, we require in addition that the degree $d_{e'}(\varphi)$ is not divisible by p for every edge e' of Σ' , and that $\varphi_{x'} : \widetilde{\mathcal{H}}(\varphi(x')) \rightarrow \widetilde{\mathcal{H}}(x')$ is separable for every vertex $x' \in V(\Sigma')$.

A tame covering of $(X, V \cup D)$ is in particular a tamely ramified morphism of curves (see Remark 4.32). In Lemma 4.33, we show that if $\varphi : (X', V' \cup D') \rightarrow (X, V \cup D)$ is a tame covering of triangulated punctured curves, then the induced map $\bar{\varphi} : \Sigma(X', V' \cup D') \rightarrow \Sigma(X, V \cup D)$ is a tame covering of metrized complexes of curves.

Lifting tame coverings of metrized complexes. The following theorem (along with the strengthening given by Theorem 7.4) is one of the main results of this paper:

Theorem 1.4. *Let $(X, V \cup D)$ be a triangulated punctured K -curve, let $\Sigma = \Sigma(X, V \cup D)$, and let $\bar{\varphi} : \Sigma' \rightarrow \Sigma$ be a tame covering of Λ -metrized complexes of k -curves. Then there exists a tame covering $\varphi : (X', V' \cup D') \rightarrow (X, V \cup D)$ of triangulated punctured K -curves inducing $\bar{\varphi}$.*

We prove a stronger version of this result in Theorem 7.4, classifying the isomorphism classes and automorphism groups of the set of liftings of Σ' . Our classification shows, in particular, that the number of lifts of Σ' to a tame covering of $(X, V \cup D)$ is *finite* (and non-zero!). This fits in with the model-theoretic point of view that an object over a non-Archimedean field should be closely controlled by objects over the residue field (the residue curves) and objects over the value group (the underlying metric graph).

We also deduce from Theorem 1.4 an *a priori* stronger-looking result, Proposition 7.13, in which we do not require that D contains the critical locus of φ . Combining Proposition 7.13 and Theorem 3.24 yields the following simply stated result:

Corollary 1.5. *Assume $\text{char}(k) = 0$ and let $\varphi : \Sigma' \rightarrow \Sigma$ be a finite harmonic morphism of Λ -metrized complexes of k -curves. Then there exists a finite morphism of triangulated punctured curves inducing φ .*

²See, however, [CKK] where a skeletal version of Liu's theorem is derived from the Liu's method of proof in the discretely valued case.

If we do not require that D contains the critical locus of φ in the setting of Theorem 1.4 (i.e., if we require φ to be a tame harmonic morphism but not necessarily a tame covering), then the set of liftings of Σ' to a tame covering of $(X, V \cup D)$ can be infinite. For example, if $\Gamma' = \{v'\}$ and $\Gamma = \{v\}$ are both points and the morphism $\varphi_{v'} : C_{v'} \cong \mathbf{P}^1 \rightarrow C_v \cong \mathbf{P}^1$ is $z \mapsto z^2$, with (X, V) a minimal triangulation of \mathbf{P}^1 and $D = \emptyset$, then there are infinitely many such lifts, corresponding to the different ways of lifting the critical points and critical values of $\varphi_{v'}$ from k to K .

Lifting tame harmonic morphisms of (augmented) metric graphs. Determining whether an effective tame harmonic morphism of augmented metric graphs is liftable to a tame harmonic morphism of metrized complexes of k -curves is equivalent to determining whether there exists a tame covering of k -curves of given genus with a prescribed ramification profile; see Proposition 8.3 for a precise statement. If $\text{char } k = 0$, this is equivalent to determining whether certain Hurwitz numbers are zero or not. Although Hurwitz numbers can be algorithmically computed, their vanishing is a subtle problem which is not yet understood. It is known that Hurwitz numbers in degree at most three are all positive, but they can be zero already in degree four.

The situation is quite different if we consider the lifting problem for finite morphisms of *non-augmented* metric graphs. In this case, there are no obstructions to lifting. The following result is an easy to state version of the more precise Theorem 8.9:

Theorem 1.6. *Any finite harmonic morphism $\bar{\varphi} : \Gamma' \rightarrow \Gamma$ of Λ -metric graphs is liftable if $\text{char}(k) = 0$.*

We stress that by definition, there is no control of the genus of the lifting curves. More precisely, we prove in Theorem 8.9 that we can prescribe the genus of the lifting of Γ , but we lose control over the genus of the lifting of Γ' .

1.7. Applications. We present here some applications to arithmetic and algebraic geometry of the lifting results described above.

Harmonic morphisms of \mathbf{Z} -metric graphs and component groups of Néron models. We discuss in (9.1) the relationship between harmonic morphisms of metric graphs and component groups of Néron models, thereby giving some applications of our lifting results to natural questions in arithmetic algebraic geometry. As a concrete example, the next proposition provides a negative answer to a question of Ribet.

Proposition 1.8. *There exists a finite morphism $f : X' \rightarrow X$ of semistable curves over a discretely valued field K_0 with $g(X) \geq 2$ such that:*

- *the special fiber of the minimal regular model of X' consists of two projective lines intersecting transversely;*
- *the induced map $f_* : \Phi_{X'} \rightarrow \Phi_X$ on component groups of Néron models is not surjective.*

Lifting tame group actions. Let \mathcal{C} be a metrized complex and let H be a finite subgroup of $\text{Aut}(\mathcal{C})$. If $\text{char}(k) = 0$, we say the action of H on \mathcal{C} is *tame* if for any vertex v of Γ , the stabilizer group H_v acts freely on a dense open subset of C_v . If $\text{char}(k) = p > 0$, we require in addition that the stabilizer subgroup H_x of H , for any point x of a curve C_v with $v \in V$, is cyclic of the form $\mathbb{Z}/d\mathbb{Z}$ for some integer d with $(d, p) = 1$. It follows from Theorem 7.4 that we can lift \mathcal{C} together with a tame action of H if and only if the quotient \mathcal{C}/H exists in the category of metrized complexes. We characterize when such a quotient exists in Theorem 9.12, of which the following result is a special case:

Theorem 1.9. *Suppose that the action of H is tame and has no isolated fixed points on the underlying metric graph of \mathcal{C} . Then there exists a smooth, proper, and geometrically connected algebraic K -curve X lifting \mathcal{C} which is equipped with an action of H commuting with the tropicalization map.*

In presence of isolated fixed points, there are additional hypothesis on the action of H to be liftable to a K -curve. As a concrete example, we prove the following characterization of all augmented tropical curves arising as the tropicalization of a hyperelliptic K -curve (see Corollary 9.17 below):

Theorem 1.10. *Let Γ be an augmented metric graph of genus $g \geq 2$ having no infinite vertices or degree one vertices of genus 0. Then there is a smooth proper hyperelliptic curve X over K of genus g having Γ as its minimal skeleton if and only if (a) there exists an involution s on Γ such that s fixes all the points $p \in \Gamma$ with $g(p) > 0$ and the quotient Γ/s is a metric tree, and (b) for every $p \in \Gamma$ the number of bridge edges adjacent to p is at most $2g(p) + 2$.*

This result can be viewed as a metric strengthening of [Cap, Theorem 4.8].

1.11. Tropical curves. A tropical curve encodes all metric graphs obtained by all possible tropicalizations of a K -curve X , i.e. by choosing all possible semistable vertex sets V and puncture sets D . Note that even if X is hyperbolic, its canonical tropicalization corresponds to the puncture set $D = \emptyset$, and hence is only one among all possible tropicalizations of X . Given a K -curve X , its tropicalization as a tropical curve is much more susceptible to retain properties of X than its tropicalization as a particular metric graph.

Tropical curves. An (augmented) Λ -tropical curve is an (augmented) Λ -metric graph considered up to tropical modifications and their inverses (see (2.27) for a precise definition).

Morphisms of tropical curves. Two harmonic morphisms of Λ -metric graphs are said to be *tropically equivalent* if there exists a harmonic morphism which is a tropical modification of each (see (2.33) for a precise definition). One makes similar definitions for morphisms of augmented metric graphs and of augmented tropical curves, with the added condition that all harmonic morphisms should be effective.

Let C' and C be (augmented) tropical curves. A *tropical morphism* $\varphi : C' \rightarrow C$ is a finite harmonic morphism of (augmented) Λ -metric graphs between some representatives of C' and C , considered up to tropical equivalence.

Gonality of tropical curves. An augmented tropical curve C is called *d-gonal* if there exists a tropical morphism of degree d from C to \mathbf{TP}^1 . By Corollary 4.28, the gonality of an augmented tropical curve is always a lower bound for the gonality of any lift to a smooth proper curve over K .

Non-sharpness of tropical gonality and ranks. We prove in Section 10 that none of the lower bounds provided by tropical ranks and gonality are sharp. For example:

Theorem 1.12.

- (1) *There exists an augmented tropical curve C of gonality 4 such that the gonality of any lifting of C is at least 5.*
- (2) *There exists an effective divisor D on a tropical curve C such that D has tropical rank equal to 1, but any effective lifting of D has rank 0.*

The construction in (1) uses the vanishing of the degree 4 Hurwitz number $H_{0,0}^4((2, 2), (2, 2), (3, 1))$. In fact we prove in Theorem 10.4 a much stronger statement: we exhibit an augmented (non-metric) graph G such that *none* of the augmented tropical curves with G as underlying augmented graph can be lifted to a 4-gonal K -curve. This means that there is a finite graph with *stable gonality* 4 (in the sense of [CKK]) which is not the (augmented) dual graph of any 4-gonal curve X/K .

The proof of (2) is based on our lifting results and an explicit example, due to Luo (see Example 10.13), of a degree 3 and rank 1 base-point free divisor D on a tropical curve C which does not appear as the fiber of any degree 3 tropical morphism from C to \mathbf{TP}^1 .

1.13. Related work. Another framework for the foundations of tropical geometry has been proposed by Kontsevich-Soibelman [KS01, KS06] and Mikhalkin [Mik06, Mik05], in which tropical objects are associated to real one-parameter families of complex varieties. We refer to [KS01, KS06] for some conjectures relating this framework to Berkovich spaces. In this setting, the notion of metrized complex of curves is similar to the notion of phase-tropical curves, and Proposition 7.13 is a consequence of Riemann's Existence Theorem. We refer the interested reader to the forthcoming paper [Mik]

for more details (see also [BBM11] where this statement is implicit, as well as Corollary 8.7 below). Tropical modifications and the “up-to-tropical-modification” point of view were introduced by Mikhalkin [Mik06].

Harmonic morphisms of finite graphs were introduced by Urakawa in [Ura00] and further explored in [BN09]. Harmonic morphisms of metric graphs have been introduced independently by several different people. Except for the integrality condition on the slopes, they appear already in Anand’s paper [Ana00]. The definition we use is the same as the one given in [Mik06, Cha12]. Concerning augmented metric graphs, the definition of effective harmonic morphisms we use is the same as in [BBM11]. The closely related, but slightly different, notion of an “indexed harmonic morphism” between weighted graphs was considered in [Cap]. The indexed pseudo-harmonic (resp. harmonic) morphisms in [Cap] are closely related to harmonic (resp. effective harmonic) morphisms in our sense when the vertex sets are fixed (see Definition 2.4(1)), and non-degenerate morphisms in the sense of [Cap] correspond to finite morphisms in our sense. One notable difference is that in [Cap], only the combinatorial type of the metric graphs are fixed; the choice of positive indices in an indexed pseudo-harmonic morphism determines the length of the edges in the source graph once the edge lengths in the target are fixed.

Finite harmonic morphisms of metrized complexes can be regarded as a metrized version of the notion of admissible cover due to Harris and Mumford [HM82] (where, in addition, arbitrary ramification above smooth points is allowed). Recall that for two semistable curves Y' and Y over a field k , a finite surjective morphism $\varphi : Y' \rightarrow Y$ is an *admissible cover* if $\varphi^{-1}(Y^{\text{sing}}) = Y'^{\text{sing}}$ and for each singular point y' of Y' , the ramification indices at y' along the two branches intersecting at y' coincide. (In addition, one usually imposes that all the other ramification of φ is simple). An admissible cover naturally gives rise to a finite harmonic morphism of metrized complexes: denoting by \mathcal{C} the regularization of Y (the metrized complex associated to Y in which each edge has length one), define \mathcal{C}' as the metrized complex obtained from Y' by letting the length of the edge associated to the double point y' be $1/r_{y'}$ (where $r_{y'}$ is the ramification index of φ at y' along either of the two branches). The morphism φ of semistable curves naturally extends to a finite harmonic morphism $\varphi : \mathcal{C}' \rightarrow \mathcal{C}$: on each edge e' of \mathcal{C}' corresponding to a double point y' of Y' , the restriction of φ to e' is linear with slope (or “expansion factor”) $r_{y'}$. Conversely, a finite harmonic morphism of metrized complexes of curves gives rise to an admissible cover of semistable curves (without the supplementary condition on simple ramification) by forgetting the metrics on both sides, remembering only the expansion factor along each edge.

Results similar to our main lifting result Theorem 7.4 have appeared in the literature. Saïdi [Sai97, Théorème 3.7] proves a version of the lifting theorem for semistable formal curves (without punctures) over a complete discrete valuation ring. His methods also make use of the tamely ramified étale fundamental group and analytic gluing arguments. Wewers [Wew99] works more generally with marked curves over a complete Noetherian local ring using deformation-theoretic arguments, proving that every tamely ramified admissible cover of marked semistable curves over the residue field of a complete Noetherian local ring lifts. Wewers classifies the possible lifts (fixing, as we do, a lift of the target) in terms of certain non-canonical “deformation data” depending on compatible choices of formal coordinates at the nodes on both the source and target curves. In addition to working over non-Noetherian valuation rings, our approach has the advantage that Wewers’ non-canonical deformation data are replaced by certain **canonical** gluing data; in particular, the automorphism group of the morphism of metrized complexes acts naturally on the set of gluing data (but not on the set of deformation data), which allows us to easily classify lifts up to isomorphism as covers of the target curve, as well as determine the automorphism group of such a lift. Our method of proof, which relies on an existence and uniqueness result for liftings of star-shaped analytic curves, seems quite natural in the context of Berkovich’s theory and is quite different from the algebraic deformation theory methods used by Wewers. Although we formulate our lifting result for (the valuation ring of) a complete and

algebraically closed valued field, it is relatively easy to descend our results to an essentially arbitrary valued field: see (7.9).

As mentioned above, we give new proofs of some results of Liu-Lorenzini [LL99], Coleman [Col03], and Liu [Liu06] concerning finite morphisms between semistable models of curves. We also provide generalizations of these results to non-discrete valuation rings, as well as “skeletal strengthenings” which are rather important from a technical standpoint. Again, our method of proof is very natural from the point of view of Berkovich’s theory and is quite different from the existing proofs. A related but somewhat different Berkovich-theoretic point of view on simultaneous semistable reduction for curves can be found in Welliaveetil’s recent preprint [Wel13]; harmonic morphisms of metrized complexes of curves play an implicit role in his paper.

In (10.1) we propose a definition for the stable gonality of a graph which coincides with the one used by Cornelissen et. al. in their recent preprint [CKK]. A slightly different notion of gonality for graphs was introduced by Caporaso in [Cap]. We also define the gonality of an augmented tropical curve, which strikes us as a more natural and perhaps more useful notion than the stable gonality of a graph (where the lengths of the edges in the source and target metric graphs are not pre-specified). We emphasize the importance of considering the dual graph of the special fiber of a semistable model of a smooth proper K -curve as an (augmented) *metric graph* and not just as a (vertex-weighted) graph. Keeping track of the natural edge lengths allows us to avoid pathological examples like Example 2.18 in [Cap] of a 3-gonal graph which is not divisorally 3-gonal.

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2. METRIC GRAPHS, TROPICAL CURVES, AND METRIZED COMPLEXES OF CURVES

In this section we recall several definitions of graphs with some additional structures and morphisms between them. A number of these definitions are now standard in tropical geometry; we refer for example to [BN07], [Mik06], [BBM11], and [AB12]. We also provide a number of examples. Some of the definitions in this section appear in the Introduction as well; we have repeated them here for the reader’s convenience.

Throughout this section, we fix Λ a non-trivial subgroup of $(\mathbf{R}, +)$.

2.1. Metric graphs. Given $r \in \mathbf{Z}_{\geq 1}$, we define $S_r \subset \mathbf{C}$ to be a “star with r branches”, i.e., a topological space homeomorphic to the convex hull in \mathbf{R}^2 of $(0, 0)$ and any r points, no two of which lie on a common line through the origin. We also define $S_0 = \{0\}$.

A *finite topological graph* Γ is the topological realization of a finite graph. That is to say, Γ is a compact 1-dimensional topological space such that for any point $p \in \Gamma$, there exists a neighborhood U_p of p in Γ homeomorphic to some S_r ; moreover there are only finitely many points p with U_p homeomorphic to S_r with $r \neq 2$.

The unique integer r such that U_p is homeomorphic to S_r is called the *valence* of p and denoted $\text{val}(p)$. A point of valence different from 2 is called an *essential vertex* of Γ . The set of *tangent directions* at p is $T_p(\Gamma) = \varinjlim_{U_p} \pi_0(U_p \setminus \{p\})$, where the limit is taken over all neighborhoods of p isomorphic to a star with r branches. The set $T_p(\Gamma)$ has $\text{val}(p)$ elements.

Definition 2.2. A *metric graph* is a finite connected topological graph Γ equipped with a complete inner metric on $\Gamma \setminus V_\infty(\Gamma)$, where $V_\infty(\Gamma) \subsetneq \Gamma$ is some (finite) set of 1-valent vertices of Γ called *infinite vertices* of Γ . (An *inner metric* is a metric for which the distance between two points x and y is the minimum of the lengths of all paths between x and y .)

A Λ -*metric graph* is a metric graph such that the distance between any two finite essential vertices of Γ lies in Λ .

A Λ -point of Γ is a point of Γ whose distance to any finite essential vertex of Γ lies in Λ .

One can equip Γ with a degenerate metric in which the infinite vertices are at infinite distance from any other point of Γ . When no confusion is possible about the subgroup Λ , we will sometimes write simply *metric graph* instead of Λ -metric graph.

Definition 2.3. Let Γ be a Λ -metric graph. A *vertex set* $V(\Gamma)$ is a finite subset of the Λ -points of Γ containing all essential vertices. An element of a fixed vertex set $V(\Gamma)$ is called a *vertex* of Γ , and the closure of a connected component of $\Gamma \setminus V(\Gamma)$ is called an *edge* of Γ . An edge which is homeomorphic to a circle is called a *loop edge*. An edge adjacent to an infinite vertex is called an *infinite edge*. We denote by $V_f(\Gamma)$ the set of finite vertices of Γ , and by $E_f(\Gamma)$ the set of finite edges of Γ .

Fix a vertex set $V(\Gamma)$. We denote by $E(\Gamma)$ the set of edges of Γ . Since Γ is a metric graph, we can associate to each edge e of Γ its length $\ell(e) \in \Lambda \cup \{+\infty\}$. Since the metric on $\Gamma \setminus V_\infty(\Gamma)$ is complete, an edge e is infinite if and only if $\ell(e) = +\infty$. The notion of vertices and edges of Γ depends, of course, on the choice of a vertex set; we will fix such a choice without comment whenever there is no danger of confusion.

Definition 2.4. Fix vertex sets $V(\Gamma')$ and $V(\Gamma)$ for two Λ -metric graphs Γ' and Γ , respectively, and let $\varphi : \Gamma' \rightarrow \Gamma$ be a continuous map.

- The map φ is called a $(V(\Gamma'), V(\Gamma))$ -*morphism of Λ -metric graphs* if we have $\varphi(V(\Gamma')) \subset V(\Gamma)$, $\varphi^{-1}(E(\Gamma)) \subset E(\Gamma')$, and the restriction of φ to any edge e' of Γ' is a dilation by some factor $d_{e'}(\varphi) \in \mathbf{Z}_{\geq 0}$.
- The map φ is called a *morphism of Λ -metric graphs* if there exists a vertex set $V(\Gamma')$ of Γ' and a vertex set $V(\Gamma)$ of Γ such that φ is a $(V(\Gamma'), V(\Gamma))$ -morphism of Λ -metric graphs.
- The map φ is said to be *finite* if $d_{e'}(\varphi) > 0$ for any edge $e' \in E(\Gamma')$.

An edge e' of Γ' is mapped to a vertex of Γ if and only if $d_{e'}(\varphi) = 0$. Such an edge is said to be *contracted* by φ . A morphism $\varphi : \Gamma' \rightarrow \Gamma$ is finite if and only if there are no contracted edges, which holds if and only if $\varphi^{-1}(p)$ is a finite set for any point $p \in \Gamma$. If φ is finite, then $p' \in V_f(\Gamma')$ if and only if $\varphi(p') \in V_f(\Gamma)$.

The integer $d_{e'}(\varphi) \in \mathbf{Z}_{\geq 0}$ in Definition 2.4 is called the *degree* of φ along e' (it is also sometimes called the *weight* of e' or *expansion factor* along e' in the literature). Since $\ell(\varphi(e')) = d_{e'}(\varphi) \cdot \ell(e')$, it follows in particular that if $d_{e'}(\varphi) \geq 1$ then e' is infinite if and only if $\varphi(e')$ is infinite. Let $p' \in V(\Gamma')$, let $v' \in T_{p'}(\Gamma')$, and let $e' \in E(\Gamma')$ be the edge in the direction of v' . We define the *directional derivative* of φ in the direction v' to be $d_{v'}(\varphi) := d_{e'}(\varphi)$. If we set $p = \varphi(p')$, then φ induces a map

$$d\varphi(p') : \{v' \in T_{p'}(\Gamma') : d_{v'}(\varphi) \neq 0\} \longrightarrow T_p(\Gamma)$$

in the obvious way.

Example 2.5. Figure 2 depicts four examples of morphisms of metric graphs $\varphi : \Gamma' \rightarrow \Gamma$. We use the following conventions in our pictures: black dots represent vertices of Γ' and Γ ; we label an edge with its degree if and only if the degree is different from 0 and 1; we do not specify the lengths of edges of Γ' and Γ .

The morphisms depicted in Figure 2(a), (b), and (c) are finite, while the one depicted in Figure 2(d) is not.

Definition 2.6. Let $\varphi : \Gamma' \rightarrow \Gamma$ be a morphism of Λ -metric graphs, let $p' \in \Gamma'$, and let $p = \varphi(p')$. The morphism φ is *harmonic at p'* provided that, for every tangent direction $v \in T_p(\Gamma)$, the number

$$d_{p'}(\varphi) := \sum_{\substack{v' \in T_{p'}(\Gamma') \\ v' \mapsto v}} d_{v'}(\varphi)$$

is independent of v . The number $d_{p'}(\varphi)$ is called the *degree of φ at p'* .

We say that φ is *harmonic* if it is *surjective* and harmonic at all $p' \in \Gamma'$; in this case the number $\deg(\varphi) = \sum_{p' \mapsto p} d_{p'}(\varphi)$ is independent of $p \in \Gamma$, and is called the *degree of φ* .

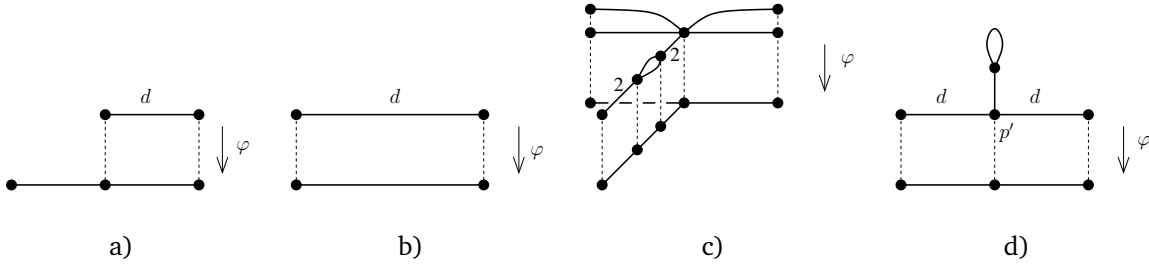


FIGURE 2.

Remark 2.7. In the above situation, if Γ consists of a single vertex v and $\varphi : \Gamma' \rightarrow \Gamma$ is a morphism of metric graphs, then φ is by definition harmonic. In this case the quantity $d_{p'}(\varphi)$ is not defined, so we include the choice of a positive integer $d_{p'}(\varphi)$ for each $p' \in V_f(\Gamma')$ as part of the data of the morphism φ . Note that if φ is finite and $\Gamma = \{p\}$, then $\Gamma' = \{p'\}$.

If both Γ' and Γ have at least one edge, then a morphism which is harmonic at all $p' \in V(\Gamma')$ is automatically surjective.

Example 2.8. The morphism depicted in Figure 2(a) is not harmonic, while the ones depicted in Figure 2(b), (c), and (d) are (for suitable choices of lengths).

2.9. For a metric graph Γ , we let $\text{Div}(\Gamma)$ denote the free abelian group on Γ . Given a harmonic morphism $\varphi : \Gamma' \rightarrow \Gamma$ of metric graphs, there are natural pull-back and push-forward homomorphisms $\varphi^* : \text{Div}(\Gamma) \rightarrow \text{Div}(\Gamma')$ and $\varphi_* : \text{Div}(\Gamma') \rightarrow \text{Div}(\Gamma)$ defined by

$$\varphi^*(p) = \sum_{p' \mapsto p} d_{p'}(\varphi) (p') \quad \text{and} \quad \varphi_*(p') = (\varphi(p'))$$

and extending linearly. It is clear that for $D \in \text{Div}(\Gamma)$ we have $\deg(\varphi^*(D)) = \deg(\varphi) \cdot \deg(D)$ and $\deg(\varphi_*(D)) = \deg(D)$.

2.10. Harmonic morphisms and harmonic functions. Given a metric graph Γ and a non-empty open set U in Γ , a function $f : U \rightarrow \mathbf{R}$ is said to be *harmonic on U* if f is piecewise affine and for all $x \in U$, the sum of the slopes of f along all outgoing tangent directions at x is equal to 0.

Since we will not use it elsewhere in the paper, we omit the proof of the following result (which is very similar to the proof of [BN09, Proposition 2.6]):

Proposition 2.11. *A morphism $\varphi : \Gamma' \rightarrow \Gamma$ of metric graphs is harmonic if and only if for every open set $U \subseteq \Gamma$ and every harmonic function $f : U \rightarrow \mathbf{R}$, the pullback function $\varphi^* f : \varphi^{-1}(U) \rightarrow \mathbf{R}$ is also harmonic.*

Equivalently, a morphism of metric graphs is harmonic if and only if germs of harmonic functions pull back to germs of harmonic functions.

2.12. Augmented metric graphs. Here we consider Λ -metric graphs together with the data of a non-negative integer at each point. For a finite point p , this integer should be thought as the genus of a “virtual algebraic curve” lying above p .

Definition 2.13. An *augmented Λ -metric graph* is a Λ -metric graph Γ along with a function $g : \Gamma \rightarrow \mathbf{Z}_{\geq 0}$, called the *genus function*, such that $g(p) = 0$ for $p \in V_\infty(\Gamma)$, and $g(p) \neq 0$ for only finitely many Λ -points $p \in \Gamma$. The *essential vertices* of Γ are the points p for which $\text{val}(p) \neq 2$ or $g(p) > 0$. A *vertex set* of Γ is a vertex set $V(\Gamma)$ of the underlying metric graph which contains all essential vertices of Γ as an augmented graph.

An augmented metric graph is said to be *totally degenerate* if the genus function is identically zero.

The *genus* of Γ is defined to be

$$g(\Gamma) = h_1(\Gamma) + \sum_{p \in \Gamma} g(p).$$

The *canonical divisor* on Γ is

$$(2.13.1) \quad K_\Gamma = \sum_{p \in \Gamma} (\text{val}(p) + 2g(p) - 2) (p).$$

A *harmonic morphism of augmented Λ -metric graphs* $\varphi : \Gamma' \rightarrow \Gamma$ is a map which is a harmonic morphism between the underlying metric graphs of Γ' and Γ .

Note that both summations in the above definition are in fact over essential vertices of Γ . The degree of the canonical divisor of an augmented Λ -metric graph Γ is $\deg(K_\Gamma) = 2g(\Gamma) - 2$. An augmented metric graph of genus 0 will also be called a *rational* augmented metric graph.

2.14. Let $\varphi : \Gamma' \rightarrow \Gamma$ be a harmonic morphism of augmented Λ -metric graphs. The *ramification divisor* of φ is the divisor $R = \sum R_{p'}(p')$, where

$$(2.14.1) \quad R_{p'} = d_{p'}(\varphi) \cdot (2 - 2g(\varphi(p'))) - (2 - 2g(p')) - \sum_{v' \in T_{p'}(\Gamma')} (d_{v'}(\varphi) - 1).$$

Note that if $p' \in V_\infty(\Gamma')$, then $R_{p'} \geq 0$ if $d_{p'}(\varphi) > 0$, and $R_{p'} = -1$ if $d_{p'}(\varphi) = 0$. The definition of $R_{p'}$ in (2.14.1) is rigged so that we have the following graph-theoretic analogue of the *Riemann-Hurwitz formula*:

$$(2.14.2) \quad K_{\Gamma'} = \varphi^*(K_\Gamma) + R.$$

In particular, the sum $R = \sum R_{p'}(p')$ is in fact finite.

Given a vertex $p' \in V(\Gamma')$ with $d_{p'}(\varphi) \neq 0$, we define the *ramification degree* of φ at p' to be $r_{p'} = R_{p'} - \#\{v' \in T_{p'}(\Gamma') \mid d_{v'}(\varphi) = 0\}$. In particular, $r_{p'} = R_{p'}$ if φ is finite.

Definition 2.15. A harmonic morphism of augmented Λ -metric graphs $\varphi : \Gamma' \rightarrow \Gamma$ is said to be *effective* if $r_{p'} \geq 0$ for every finite vertex p' of Γ' with $d_{p'}(\varphi) \neq 0$.

Remark 2.16. Among all harmonic morphisms of augmented metric graphs, only the effective ones have a chance to be lifted to an algebraic map between algebraic curves (they should also have a finite representative, see (2.27)). The condition $r_{p'} \geq 0$ comes from the Riemann-Hurwitz formula in algebraic geometry, and its necessity will become transparent when we define metrized complexes of curves and harmonic morphisms between them; see Remark 2.23.

Definition 2.17. Let $\varphi : \Gamma' \rightarrow \Gamma$ be a finite harmonic morphism of augmented Λ -metric graphs. We say that φ is *étale* at a point $p' \in \Gamma'$ provided that $R_{p'} = 0$. We say that φ is *generically étale* if R is supported on the set of infinite vertices of Γ' and that φ is *étale* if $R = 0$.

Note that a generically étale morphism of augmented metric graphs is effective.

Example 2.18. All morphisms depicted in Figure 2(b), (c), and (d), are effective provided that all the target graphs are totally degenerate. If the source graph is also totally degenerate and all 1-valent vertices are infinite vertices, then the morphisms depicted in Figure 2(b) and (c) are generically étale.

Suppose that all 1-valent vertices are infinite vertices in Figure 3, and that $g(p) = 0$ in Figure 3(a) and $g(p) = 1$ in Figure 3(b). Then $r_{p'} = 2g(p') - 1$ and $r_{p'_i} = 2g(p'_i) - 2$, so the morphism depicted in 3(a) is effective if and only if $g(p') \geq 1$, and the morphism depicted in 3(b) is effective if and only if both vertices p'_1 and p'_2 have genus at least 1.

2.19. Metrized complexes of curves. Recall that k is an algebraically closed field. A metrized complex of curves is, roughly speaking, an augmented metric graph Γ together with a chosen vertex set $V(\Gamma)$ and a marked algebraic k -curve of genus $g(p)$ for each finite vertex $p \in V(\Gamma)$. More precisely:

Definition 2.20. A *Λ -metrized complex of k -curves* consists of the following data:

- An augmented Λ -metric graph Γ equipped with the choice of a distinguished vertex set $V(\Gamma)$.
- For every finite vertex p of Γ , a smooth, proper, connected k -curve C_p of genus $g(p)$.
- An injective function $\text{red}_p : T_p(\Gamma) \rightarrow C_p(k)$, called the *reduction map*. We call the image of red_p the set of *marked points* on C_p .

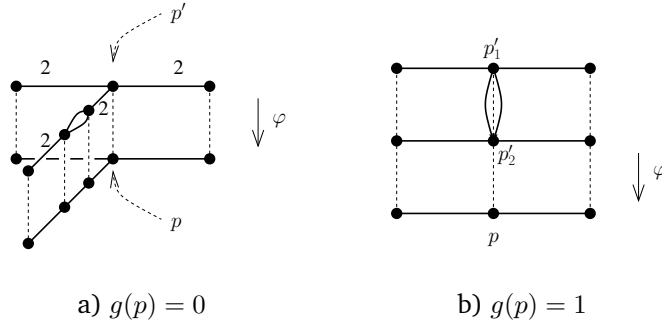
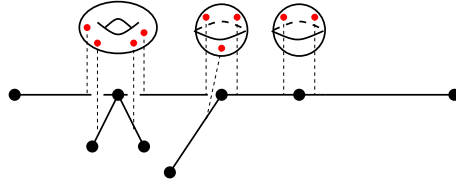


FIGURE 3.

Example 2.21. Figure 2.21 depicts a particular metrized complex of curves over \mathbb{C} . At each finite vertex p there is an associated compact Riemann surface C_p , together with a finite set of marked points (in red).



Definition 2.22. A harmonic morphism of metrized complexes of curves consists of a harmonic $(V(\Gamma'), V(\Gamma))$ -morphism $\varphi : \Gamma' \rightarrow \Gamma$ of augmented metric graphs, and for every finite vertex p' of Γ' with $d_{p'}(\varphi) > 0$ a finite morphism of curves $\varphi_{p'} : C'_{p'} \rightarrow C_{\varphi(p')}$, satisfying the following compatibility conditions:

- (1) For every finite vertex $p' \in V(\Gamma')$ and every tangent direction $v' \in T_{p'}(\Gamma')$ with $d_{v'}(\varphi) > 0$, we have $\varphi_{p'}(\text{red}_{p'}(v')) = \text{red}_{\varphi(p')}(d\varphi(p')(v'))$, and the ramification degree of $\varphi_{p'}$ at $\text{red}_{p'}(v')$ is equal to $d_{v'}(\varphi)$.
- (2) For every finite vertex $p' \in V(\Gamma')$ with $d_{p'}(\varphi) > 0$, every tangent direction $v \in T_{\varphi(p')}(\Gamma)$, and every point $x' \in \varphi_{p'}^{-1}(\text{red}_{\varphi(p')}(v)) \subset C'_{p'}(k)$, there exists $v' \in T_{p'}(\Gamma')$ such that $\text{red}_{p'}(v') = x'$.
- (3) For every finite vertex $p' \in V(\Gamma')$ with $d_{p'}(\varphi) > 0$ we have $d_{p'}(\varphi) = \deg(\varphi_{p'})$.

A harmonic morphism of metrized complexes of curves is called *finite* if the underlying harmonic morphism of (augmented) metric graphs is finite. Metrized complexes of curves with harmonic morphisms between them form a category, with finite harmonic morphisms giving rise to a subcategory.

If Γ has at least one edge, then (3) follows from (1) and (2), since the sum of the ramification degrees of a finite morphism of curves along any fiber is equal to the degree of the morphism.

Remark 2.23. It follows from Riemann-Hurwitz formula applied to the maps $\varphi_{p'} : C'_{p'} \rightarrow C_{\varphi(p')}$ that a harmonic morphism of metrized complexes of curves underlies an *effective* harmonic morphism of augmented metric graphs.

Definition 2.24. Let $\varphi : \mathcal{C}' \rightarrow \mathcal{C}$ be a finite harmonic morphism of metrized complexes of curves. We say that φ is a *tame harmonic morphism* if either $\text{char}(k) = 0$ or $\text{char}(k) = p > 0$ and

- (1) $d_{e'}(\varphi)$ is prime to p for all edges e' of Γ' .
- (2) $\varphi_{p'}$ is tamely ramified for all finite vertices $p' \in V(\Gamma')$.

We call φ a *tame covering* if in addition

- (3) $\varphi : \Gamma' \rightarrow \Gamma$ is a generically étale finite morphism of augmented metric graphs.

Note that being generically étale in (3) means that all the ramification points of the maps $\varphi_{p'} : C'_{p'} \rightarrow C_{\varphi(p')}$ are among the marked points of $C'_{p'}$; in particular, the ramification indices at these points are “visible” in the underlying morphism of metric graphs as the degrees of φ along the various tangent directions at p' .

Proposition 2.25. *Let $\varphi : C' \rightarrow C$ be a tame covering of metrized complexes of curves, let p' be a finite vertex of Γ' , and suppose that $\varphi_{p'}$ is ramified at a point $x' \in C'_{p'}(k)$. Then there exists $v' \in T_{p'}(\Gamma')$ such that $\text{red}_{p'}(v') = x'$. In particular, $\varphi_{p'}$ is tamely ramified.*

Proof. Let $p = \varphi(p')$. Since φ is generically étale we have

$$0 = R_{p'} = d_{p'}(\varphi)(2 - 2g(p)) - (2 - 2g(p')) - \sum_{v' \in T_{p'}(\Gamma')} (d_{v'}(\varphi) - 1).$$

Since $g(p) = g(C_p)$, $g(p') = g(C'_{p'})$, and $d_{p'}(\varphi)$ is the degree of $\varphi_{p'} : C'_{p'} \rightarrow C_p$, this gives

$$\sum_{v' \in T_{p'}(\Gamma')} (d_{v'}(\varphi) - 1) = \deg(\varphi_{p'})(2 - 2g(C_p)) - (2 - 2g(C'_{p'})).$$

The Riemann-Hurwitz formula as applied to $\varphi_{p'}$ says that the right-hand side of the above equation is equal to the degree of the ramification divisor of $\varphi_{p'}$ (this holds whenever $\varphi_{p'}$ is separable), so the proposition follows because for all $v' \in T_{p'}(\Gamma')$, the ramification degree of $\varphi_{p'}$ at $\text{red}_{p'}(v')$ is $d_{v'}(\varphi)$. ■

Remark 2.26. Loosely speaking, Proposition 2.25 says that all ramification of all morphisms $\varphi_{p'}$ is “visible” in the edge degrees of the underlying morphism of metric graphs. It follows from Proposition 2.25 that if $\Gamma' = \{p'\}$, then $\varphi_{p'}$ is étale.

2.27. Tropical modifications and tropical curves. Here we introduce an equivalence relation among metric graphs; an equivalence class for this relation will be called a *tropical curve*.

Definition 2.28. An *elementary tropical modification* of a Λ -metric graph Γ_0 is a Λ -metric graph $\Gamma = [0, +\infty] \cup \Gamma_0$ obtained from Γ_0 by attaching the segment $[0, +\infty]$ to Γ_0 in such a way that $0 \in [0, +\infty]$ gets identified with a finite Λ -point $p \in \Gamma_0$. If Γ_0 is augmented, then Γ naturally inherits a genus function from Γ_0 by declaring that every point of $(0, +\infty]$ has genus 0.

An (augmented) Λ -metric graph Γ obtained from an (augmented) Λ -metric graph Γ_0 by a finite sequence of elementary tropical modifications is called a *tropical modification* of Γ_0 .

If Γ is a tropical modification of Γ_0 , then there is a natural retraction map $\tau : \Gamma \rightarrow \Gamma_0$ which is the identity on Γ_0 and contracts each connected component of $\Gamma \setminus \Gamma_0$ to the unique point in Γ_0 lying in the topological closure of that component. The map τ is a (non-finite) harmonic morphism of (augmented) metric graphs.

Example 2.29. We depict an elementary tropical modification in Figure 4(a), and a tropical modification which is the sequence of two elementary tropical modifications in Figure 4(b).

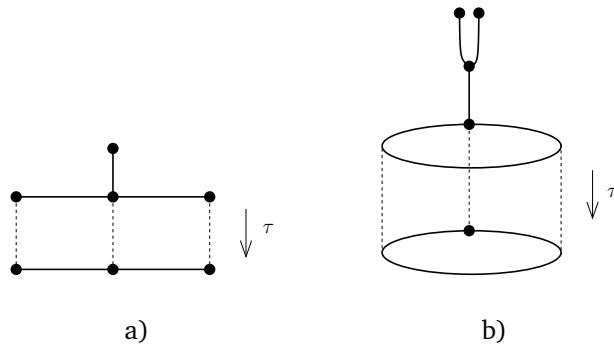


FIGURE 4. Two tropical modifications

Tropical modifications generate an equivalence relation \sim on the set of (augmented) Λ -metric graphs.

Definition 2.30. A Λ -tropical curve (resp. an augmented Λ -tropical curve) is an equivalence class of Λ -metric graphs (resp. augmented Λ -metric graphs) with respect to \sim .

In other words, a Λ -tropical curve is a Λ -metric graph considered up to tropical modifications and their inverses (and similarly for augmented tropical curves). By abuse of terminology, we will often refer to a tropical curve in terms of one of its metric graph representatives.

Example 2.31. There exists a unique rational (augmented) tropical curve, which we denote by TP^1 . Any rational (augmented) metric graph whose 1-valent vertices are all infinite is obtained by a sequence of tropical modifications from the metric graph consisting of a unique finite vertex (of genus 0).

Example 2.32. Let Γ_0 be a Λ -metric graph, $p \in \Gamma_0$ a finite Λ -point, and $l \in \Lambda \setminus \{0\}$. We can construct a new Λ -metric graph Γ by attaching the segment $[0, l]$ to Γ_0 via the identification of $0 \in [0, l]$ with p . Then Γ_0 and Γ represent the same tropical curve, since the elementary tropical modification of Γ_0 at p and the elementary tropical modification of Γ at the right-hand endpoint of $[0, l]$ are the same metric graph.

Definition 2.33. Let Γ (resp. Γ') be a representative of a Λ -tropical curve C (resp. C'), and assume we are given a harmonic morphism of Λ -metric graphs $\varphi : \Gamma' \rightarrow \Gamma$.

An *elementary tropical modification* of φ is a harmonic morphism $\varphi_1 : \Gamma'_1 \rightarrow \Gamma_1$ of Λ -metric graphs, where $\tau : \Gamma_1 \rightarrow \Gamma$ is an elementary tropical modification, $\tau' : \Gamma'_1 \rightarrow \Gamma'$ is a tropical modification, and such that $\varphi \circ \tau' = \tau \circ \varphi_1$.

A *tropical modification* of φ is a finite sequence of elementary tropical modifications of φ .

Two harmonic morphisms φ_1 and φ_2 of Λ -metric graphs are said to be *tropically equivalent* if there exists a harmonic morphism which is a tropical modification of both φ_1 and φ_2 .

A *tropical morphism of tropical curves* $\varphi : C' \rightarrow C$ is a harmonic morphism of Λ -metric graphs between some representatives of C' and C , considered up to tropical equivalence, and which has a finite representative.

One makes similar definitions for morphisms of augmented tropical curves, with the additional condition that all harmonic morphisms should be effective.

Remark 2.34. The collection of Λ -metric graphs (resp. augmented Λ -metric graphs), together with harmonic morphisms (resp. effective harmonic morphisms) between them, forms a category. Except for the condition of having a finite representative, one could try to think of tropical curves, together with tropical morphisms between them, as the localization of this category with respect to tropical modifications. However, there are some technical problems which arise when one tries to make this rigorous (at least if we demand that the localized category admit a calculus of fractions): as we will see in Example 2.35, tropical equivalence is *not* a transitive relation between morphisms of Λ -metric graphs. On the other hand, the restriction of tropical equivalence of morphisms (resp. of augmented morphisms) to the collection of finite morphisms (resp. of generically étale morphisms) is transitive (and hence an equivalence relation). This is one reason why we include the condition that φ has a finite representative in our definition of a morphism of tropical curves; another reason is that all morphisms of tropical curves which arise from algebraic geometry automatically satisfy this condition.

Example 2.35. The morphism of (totally degenerate augmented) metric graphs depicted in Figure 2(c) (resp. 5(b)) is an elementary tropical modification of the one depicted in 5(a) (resp. 2(c)).

The tropical morphisms φ_1 and φ_2 of totally degenerate augmented tropical curves depicted in Figure 5(c) and (d) are both elementary tropical modifications of the morphism φ depicted in Figure 5(e).

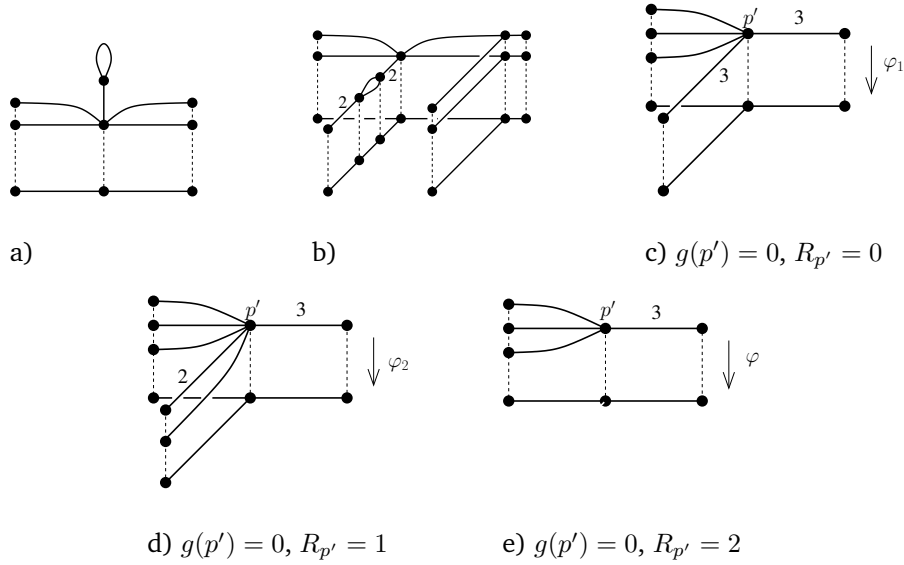


FIGURE 5.

The tropical morphisms φ_1 and φ_2 depicted in Figure 6(a) and (b) are both elementary tropical modifications of the morphism φ depicted in Figure 6(c).

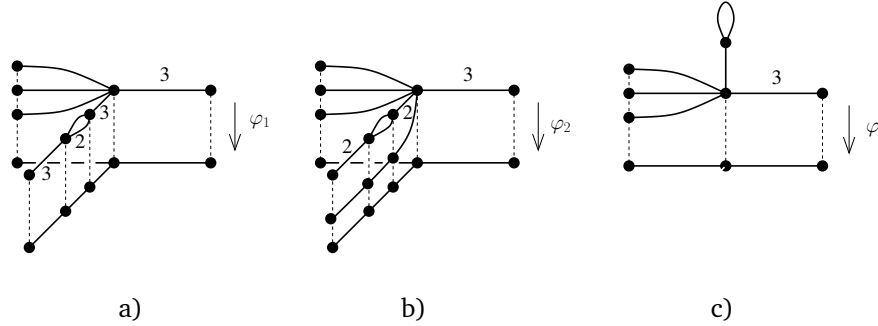


FIGURE 6.

On the other hand, the harmonic morphism $\varphi : \Gamma' \rightarrow \Gamma$ depicted in Figure 2(d) with $d = 1$ is not tropically equivalent to any finite morphism: since φ has degree 1, the cycle of the source graph will be contracted to a point by any harmonic morphism of metric graphs tropically equivalent to φ . In particular, φ does not give rise to a tropical morphism.

As mentioned above, tropical equivalence is not transitive among morphisms of metric graphs (resp. of augmented metric graphs). For example, the two morphisms φ_1 and φ_2 depicted in Figure 5(c) and (d) are not tropically equivalent as augmented morphisms: since $R_{p'} = 0$ in Figure 5(c), any edge appearing in a tropical modification of φ_1 will have degree 1.

Note that the preceding harmonic morphisms φ_1 and φ_2 are tropically equivalent as morphisms of metric graphs (i.e. forgetting the genus function). However, tropical equivalence is not transitive for tropical morphisms either, for essentially the same reason: the two tropical morphisms φ_1 and φ_2 depicted in Figure 6(a) and (b) are not tropically equivalent.

Nevertheless, the restriction of tropical equivalence of morphisms to the set of finite morphisms (resp. generically étale morphisms) is an equivalence relation. Hence a tropical morphism (resp. an

augmented tropical morphism) can also be thought as an equivalence class of finite harmonic morphisms (resp. generically étale morphisms). In particular there exists a natural composition rule for tropical morphisms (resp. augmented tropical morphisms), turning tropical curves (resp. augmented tropical curves) equipped with tropical morphisms into a category.

Remark 2.36. In the definition of a tropical morphism of augmented tropical curves, in addition to the condition of being a harmonic morphism and the “up to tropical modifications” considerations, we imposed two rather strong conditions, namely being effective and having a finite representative. We already saw in Example 2.35 that the finiteness condition is non-trivial. The effectiveness condition is also non-trivial: for example, the harmonic morphism $\varphi : \Gamma' \rightarrow \Gamma$ of totally degenerate augmented metric graphs depicted in Figure 2(d) with $d = 2$ is not tropically equivalent to any finite effective morphism of totally degenerate augmented metric graphs. Indeed, for any tropical modification of φ which is effective, at most two edges adjacent to p' can have degree 2; since Γ' already has two such edges for φ , any tropical modification of φ which is finite and effective will contract the cycle of Γ' to a point.

We refer to [BM] for a general definition of a tropical morphism $\varphi : C \rightarrow X$ from an augmented tropical curve to a non-singular tropical variety, including Definition 2.33 as a particular case.

We will also make use in the text of the notion of tropical modifications of metrized complexes of curves.

Definition 2.37. Let C_0 be a Λ -metrized complex of k -curves.

- A *refinement* of C_0 is any Λ -metrized complex of k -curves C obtained from C_0 by adding a finite set of Λ -points S of $C_0 \setminus V(C_0)$ to the set $V(C_0)$ of vertices of C_0 (see Definition 2.20), setting $C_p = \mathbf{P}_k^1$ for all $p \in S$, and defining the map red_p by choosing any two distinct closed points of C_p .
- An *elementary tropical modification* of C_0 is a Λ -metrized complex of k -curves C obtained from C_0 by an elementary tropical modification of the underlying metric graph at a vertex p of C , with the map red_p extended to e by choosing any closed point of C_p not in the image of the reduction map for C_0 .
- Any metrized complex of curves C obtained from a metrized complex of curves C_0 by a finite sequence of refinements and elementary tropical modifications is called a *tropical modification* of C_0 .

3. ANALYTIC CURVES AND THEIR SKELETA

In this section we recall the definition and basic properties of the skeleton of the analytification of an algebraic curve as outlined in [Tem] and elaborated in [BPR11, §5]. These foundational results are well-known to experts, and appear in various guises in the literature. See for example Berkovich [Ber90, Ber99], Thuillier [Thu05], and Ducros [Duc08]. In this section we also show that the skeleton is naturally a metrized complex of curves, and prove that any metrized complex arises as the skeleton of a curve. Finally, we introduce the important notion of a triangulated punctured curve, which is essentially a punctured curve along with the data of a skeleton with a distinguished set of vertices.

Recall that K is an algebraically closed field which is complete with respect to a nontrivial non-Archimedean valuation $\text{val} : K \rightarrow \mathbf{R} \cup \{\infty\}$. Its valuation ring is R , its maximal ideal is \mathfrak{m}_R , its residue field is $k = R/\mathfrak{m}_R$, and its value group is $\Lambda = \text{val}(K^\times) \subset \mathbf{R}$.

By an *analytic curve* we mean a strictly K -analytic space X of pure dimension 1. For our purposes the most important example of an analytic curve is the analytification of a smooth, connected, projective algebraic K -curve. If X is an analytic curve, then we define $\mathbf{H}(X) = X \setminus X(K)$ as in [BPR11, 3.10].

3.1. Analytic building blocks. We identify the set underlying the analytification of the affine line with the set of multiplicative seminorms $\|\cdot\| : K[t] \rightarrow \mathbf{R}_{\geq 0}$ which extend the absolute value on K .

We let $\text{val} : \mathbf{A}^{1,\text{an}} \rightarrow \mathbf{R} \cup \{\infty\}$ denote the valuation map

$$\text{val}(x) = -\log |x| = -\log \|t\|_x.$$

For $a \in K^\times$ the *standard closed ball of radius $|a|$* is the affinoid domain

$$\mathbf{B}(a) = \text{val}^{-1}([\text{val}(a), \infty]) \subset \mathbf{A}^{1,\text{an}}$$

and the *standard open ball of radius $|a|$* is the open analytic domain

$$\mathbf{B}(a)_+ = \text{val}^{-1}((\text{val}(a), \infty]) \subset \mathbf{A}^{1,\text{an}}.$$

Note that scaling by a gives isomorphisms $\mathbf{B}(1) \xrightarrow{\sim} \mathbf{B}(a)$ and $\mathbf{B}(1)_+ \xrightarrow{\sim} \mathbf{B}(a)_+$. We call $\mathbf{B}(1)$ (resp. $\mathbf{B}(1)_+$) the *standard closed* (resp. *open*) *unit ball*. For $a \in K^\times$ with $\text{val}(a) \geq 0$ the *standard closed annulus of modulus $\text{val}(a)$* is the affinoid domain

$$\mathbf{S}(a) = \text{val}^{-1}([0, \text{val}(a)]) \subset \mathbf{A}^{1,\text{an}}$$

and if $\text{val}(a) > 0$ the *standard open annulus of modulus $\text{val}(a)$* is the open analytic domain

$$\mathbf{S}(a)_+ = \text{val}^{-1}((0, \text{val}(a))) \subset \mathbf{A}^{1,\text{an}}.$$

The *standard punctured open ball* is the open analytic domain

$$\mathbf{S}(0)_+ = \text{val}^{-1}((0, \infty)) = \mathbf{B}(1)_+ \setminus \{0\} \subset \mathbf{A}^{1,\text{an}}.$$

By a *closed unit ball* (resp. *open unit ball*, resp. *closed annulus*, resp. *open annulus*, resp. *punctured open ball*) we will mean a K -analytic space which is isomorphic to the standard closed unit ball (resp. the standard open unit ball, resp. the standard closed annulus of modulus $\text{val}(a)$ for some $a \in R \setminus \{0\}$, resp. the standard open annulus of modulus $\text{val}(a)$ for some $a \in \mathfrak{m}_R \setminus \{0\}$, resp. the standard punctured open ball). A *generalized open annulus* is a K -analytic space which is either an open annulus or a punctured open ball. It is a standard fact that the modulus of a closed (resp. open) annulus is an isomorphism invariant; see for instance [BPR11, Corollary 5.6]. We define the modulus of a punctured open ball to be ∞ .

Let $a \in \mathfrak{m}_R$ (so $a = 0$ is allowed). There is a natural section $\sigma : (0, \text{val}(a)) \rightarrow \mathbf{S}(a)_+$ of the valuation map $\text{val} : \mathbf{S}(a)_+ \rightarrow (0, \text{val}(a))$ sending r to the maximal point of the affinoid domain $\text{val}^{-1}(r)$ if $r \in \Lambda$, resp. the only point of $\text{val}^{-1}(r)$ if $r \notin \Lambda$. The *skeleton* of $\mathbf{S}(a)_+$ is defined to be the image of σ and is denoted $\Sigma(\mathbf{S}(a)_+) = \sigma((0, \text{val}(a)))$. We will always identify the skeleton of the standard generalized open annulus $\mathbf{S}(a)_+$ with the interval or ray $(0, \text{val}(a))$. It follows from [BPR11, Proposition 5.5] that the skeleton of a standard open annulus or standard punctured open ball is an isomorphism invariant, so if A is a generalized open annulus, then we can define the *skeleton* $\Sigma(A)$ of A to be the image of the skeleton of a standard open annulus or standard punctured open ball $\mathbf{S}(a)_+$ under any isomorphism $\mathbf{S}(a)_+ \xrightarrow{\sim} A$.

Let T be a metric space and let $x, y \in T$. A *geodesic segment from x to y* is the image of a locally isometric embedding $[a, b] \hookrightarrow T$ with $a \mapsto x$ and $b \mapsto y$. We often identify a geodesic segment with its image in T . Recall that an **R-tree** is a metric space T with the following properties:

- (i) For all $x, y \in T$ there is a unique geodesic segment $[x, y]$ from x to y .
- (ii) For all $x, y, z \in T$, if $[x, y] \cap [y, z] = \{y\}$, then $[x, z] = [x, y] \cup [y, z]$.

See [BR10, Appendix B]. It is proved in §1.4 of loc. cit. that $\mathbf{H}(\mathbf{B}(1))$ is naturally an **R-tree**. Since any path-connected subspace of an **R-tree** is an **R-tree** as well, if X is a standard open annulus or standard (punctured) open ball then $\mathbf{H}(X)$ is an **R-tree**. It is proved in [BPR11, Corollary 5.61] that the metric structure on $\mathbf{H}(X)$ is an isomorphism invariant, so the same statement applies to open balls and generalized open annuli. For $a \in \mathfrak{m}_R$ the section $\sigma : (0, \text{val}(a)) \rightarrow \mathbf{S}(a)_+$ is an isometry.

It also follows from the results in [BR10, §1.4] that for any $x \in \mathbf{H}(\mathbf{B}(1))$ and any type-1 point $y \in \mathbf{B}(1)$, there is a unique continuous injection $\alpha : [0, \infty] \hookrightarrow \mathbf{B}(1)$ with $\alpha(0) = x$ and $\alpha(\infty) = y$, $\alpha([0, \infty)) \subset \mathbf{H}(\mathbf{B}(1))$, and such that α is an isometry when restricted to $[0, \infty)$. We let $[x, y]$ denote the image of α and we call $[x, y]$ the *geodesic segment from x to y* . Similarly, if x and y are both type-1 points then there is a unique continuous injection $\alpha : [-\infty, \infty] \rightarrow \mathbf{B}(1)$ with $\alpha(-\infty) = x$ and

$\alpha(\infty) = y$, $\alpha(\mathbf{R}) \subset \mathbf{H}(\mathbf{B}(1))$, and such that $\alpha|_{\mathbf{R}}$ is an isometry; the image of α is called the *geodesic segment from x to y* and is denoted $[x, y]$. Restricting to a suitable analytic subdomain of $\mathbf{B}(1)$ allows us to define geodesic segments between any two points of an open ball or generalized open annulus.

3.2. Open balls. The closure of $\mathbf{B}(1)_+$ in $\mathbf{B}(1)$ consists of $\mathbf{B}(1)_+$ and a single type-2 point x , called the *end* of $\mathbf{B}(1)_+$. The end is the Shilov boundary point of $\mathbf{B}(1)$: see for instance the proof of [BPR11, Lemma 5.16]. The valuation map $\text{val} : \mathbf{B}(1)_+ \rightarrow (0, \infty]$ extends to a continuous map $\text{val} : \mathbf{B}(1)_+ \cup \{x\} \rightarrow [0, \infty]$. The set $\mathbf{B}(1)_+ \cup \{x\}$ is path-connected and compact, being a closed subspace of the compact space $\mathbf{B}(1)$. (In fact $\mathbf{B}(1)_+ \cup \{x\}$ is the one-point compactification of $\mathbf{B}(1)_+$.) For any $y \in \mathbf{B}(1)_+$ the geodesic segment $[x, y] \subset \mathbf{B}(1)$ is contained in $\mathbf{B}(1)_+ \cup \{x\}$.

Lemma 3.3. *Let $X = \text{Spec}(A)$ be an irreducible affine K -curve and let $\varphi : \mathbf{B}(1)_+ \rightarrow X^{\text{an}}$ be a morphism with finite fibers. Then φ extends in a unique way to a continuous map $\mathbf{B}(1)_+ \cup \{x\} \rightarrow X^{\text{an}}$, and the image of x is a type-2 point of X^{an} .*

Proof. Let $f \in A$ be nonzero and define $F : (0, \infty) \rightarrow \mathbf{R}$ by $F(r) = -\log |f \circ \varphi(\sigma(r))|$, where $\sigma : (0, \infty) \rightarrow \mathbf{S}(0)_+ \subset \mathbf{B}(1)_+$ is the canonical section of val . By [BPR11, Proposition 5.10], F is a piecewise affine function which is differentiable away from finitely many points $r \in \Lambda \cap (0, \infty)$, and for any point $r \in (0, \infty)$ at which F is differentiable, the derivative of F is equal to the number of zeros y of $f \circ \varphi$ with $\text{val}(y) > r$. It follows that $F(\Lambda \cap (0, \infty)) \subset \Lambda$. Since $f \circ \varphi$ has finitely many zeros, the limit $\lim_{r \rightarrow 0} F(r)$ exists and is contained in Λ . We define $\|f\|_0 = \exp(-\lim_{r \rightarrow 0} F(r))$. It is easy to see that $f \mapsto \|f\|_0$ is a multiplicative seminorm on A extending the absolute value on K , so $\|\cdot\|_0$ is a point in X^{an} . Define $\varphi(x) = \|\cdot\|_0$. One shows as in the proof of [BPR11, Lemma 5.16] that φ is continuous on $\mathbf{B}(1)_+ \cup \{x\}$.

It remains to show that $\varphi(x) = \|\cdot\|_0$ is a type-2 point of X^{an} . Let $f \in A$ be a non-constant function that has a zero on $\varphi(\mathbf{B}(1)_+)$. Since $-\log \|f\|_0 \in \Lambda$, there exists $\alpha \in K^\times$ such that $\|\alpha f\|_0 = 1$; replacing f by αf , we may and do assume that $\|f\|_0 = 1$. Since $\varphi \circ f$ has a zero, by the above we have that $F = -\log |f \circ \varphi \circ \sigma|$ is monotonically increasing, so $F(r) > 0$ for all $r \in (0, \infty)$. For any $r \in (0, \infty)$, the maximal point of $\text{val}^{-1}(r) \subset \mathbf{B}(1)_+$ is equal to $\sigma(r)$, so for any $y \in \mathbf{B}(1)_+$ such that $\text{val}(y) = r$ we have $F(r) \leq -\log |f \circ \varphi(y)|$. It follows that $|f \circ \varphi(y)| < 1$ for all $y \in \mathbf{B}(1)_+$, so $\varphi(f(\mathbf{B}(1)_+)) \subset \mathbf{B}(1)_+$. Since $\mathbf{B}(1)_+$ is dense in $\mathbf{B}(1)_+ \cup \{x\}$, the image of x under $\varphi \circ f$ is contained in the closure $\mathbf{B}(1)_+ \cup \{x\}$ of $\mathbf{B}(1)_+$ in $\mathbf{B}(1)$. Since $1 = \|f\|_0 = |f \circ \varphi(x)|$, the point $f \circ \varphi(x) \notin \mathbf{B}(1)_+$, so $f \circ \varphi(x) = x$, which is a type-2 point of $\mathbf{A}^{1, \text{an}}$. Therefore $\varphi(x)$ is a type-2 point of X^{an} . ■

Applying Lemma 3.3 to a morphism $\varphi : \mathbf{B}(1)_+ \rightarrow \mathbf{B}(1)_+ \subset \mathbf{A}^{1, \text{an}}$, we see that any automorphism of $\mathbf{B}(1)_+$ extends (uniquely) to a homeomorphism $\mathbf{B}(1)_+ \cup \{x\} \rightarrow \mathbf{B}(1)_+ \cup \{x\}$, so it makes sense to speak of the *end* of any open ball. If B is an open ball with end x , we let \overline{B} denote $B \cup \{x\}$.

Let B be an open ball with end x . We define a partial ordering on \overline{B} by declaring $y \leq z$ if $y \in [x, z]$; again see [BR10, §1.4]. Equivalently, for $y, z \in B$ we have $y \leq z$ if and only if $|f(y)| \geq |f(z)|$ for all analytic functions f on B . The following lemma is proved as in [BPR11, Proposition 5.27].

Lemma 3.4. *Let B be an open ball and let $y \in B$ be a type-2 point. Then $B \setminus \{y\}$ is a disjoint union of the open annulus $A = \{z \in B : z \not\leq y\}$ with infinitely many open balls, and $B_y = \{z \in B : z \geq y\}$ is an affinoid subdomain of B isomorphic to the closed ball $\mathbf{B}(1)$.*

3.5. Open annuli. Let $a \in \mathfrak{m}_R \setminus \{0\}$. The closure of $\mathbf{S}(a)_+$ in $\mathbf{B}(1)$ consists of $\mathbf{S}(a)_+$ and the two type-2 points $x = \sigma(0)$ and $y = \sigma(\text{val}(a))$, called the *ends* of $\mathbf{S}(a)_+$: again see the proof of [BPR11, Lemma 5.16]. The valuation map $\text{val} : \mathbf{S}(a)_+ \rightarrow (0, \text{val}(a))$ extends to a continuous map $\text{val} : \mathbf{S}(a)_+ \cup \{x, y\} \rightarrow [0, \text{val}(a)]$, and for any $z \in \mathbf{S}(a)_+$ the geodesic segments $[x, z]$ and $[y, z]$ are contained in $\mathbf{S}(a)_+ \cup \{x, y\}$. The following lemma is proved in the same way as the first part of Lemma 3.3.

Lemma 3.6. *Let $X = \text{Spec}(A)$ be an irreducible affine K -curve, let $a \in \mathfrak{m}_R \setminus \{0\}$, and let $\varphi : \mathbf{S}(a)_+ \rightarrow X^{\text{an}}$ be a morphism with finite fibers. Let $x = \sigma(0)$ and $y = \sigma(\text{val}(a))$ be the ends of $\mathbf{S}(a)_+$. Then φ extends in a unique way to a continuous map $\mathbf{S}(a)_+ \cup \{x, y\} \rightarrow X^{\text{an}}$.*

It follows from Lemma 3.6 that any automorphism of $\mathbf{S}(a)_+$ extends to a homeomorphism $\mathbf{S}(a)_+ \cup \{x, y\} \xrightarrow{\sim} \mathbf{S}(a)_+ \cup \{x, y\}$, so it makes sense to speak of the *ends* of any open annulus. If A is an open annulus with ends x, y , then we let \bar{A} denote the compact space $A \cup \{x, y\}$.

The closure of the punctured open ball $\mathbf{S}(0)_+$ in $\mathbf{B}(1)$ is equal to $\mathbf{S}(0)_+ \cup \{0, x\}$, where x is the end of $\mathbf{B}(1)_+$. We define x to be the *end* of $\mathbf{S}(0)_+$ and 0 to be the *puncture*. As above, the end and puncture of $\mathbf{S}(0)_+$ are isomorphism invariants, so it makes sense to speak of the *end* and the *puncture* of any punctured open ball. If A is a punctured open ball with end x and puncture y , then we let \bar{A} denote the compact space $A \cup \{x, y\}$.

Let A be a generalized open annulus. Each connected component of $A \setminus \Sigma(A)$ is an open ball [BPR11, Lemma 5.12], and if B is such a connected component with end x , then the inclusion $B \hookrightarrow A$ extends to an inclusion $\bar{B} \hookrightarrow A$ with x mapping into $\Sigma(A)$; the image of \bar{B} is the closure of B in A . We define the retraction to the skeleton $\tau : A \rightarrow \Sigma(A)$ by fixing $\Sigma(A)$ and sending each connected component of $A \setminus \Sigma(A)$ to its end. If $A = \mathbf{S}(a)_+$, then the retraction τ coincides with $\sigma \circ \text{val} : \mathbf{S}(a)_+ \rightarrow \Sigma(\mathbf{S}(a)_+)$; in particular, if $(r, s) \subset (0, \text{val}(a)) = \mathbf{S}(a)_+$, then $\tau^{-1}((r, s)) = \text{val}^{-1}((r, s))$.

3.7. The skeleton of a curve. Let X be a smooth, connected, proper algebraic K -curve and let $D \subset X(K)$ be a finite set of closed points. The set $\mathbf{H}(X^{\text{an}})$ is naturally a metric space [BPR11, Corollary 5.62], although the metric topology on $\mathbf{H}(X^{\text{an}})$ is much finer than the topology induced by the topology on the K -analytic space X^{an} . The metric on $\mathbf{H}(X^{\text{an}})$ is locally modeled on an \mathbf{R} -tree [BPR11, Proposition 5.63].

Definition 3.8. A *semistable vertex set* of X is a finite set V of type-2 points of X^{an} such that $X^{\text{an}} \setminus V$ is a disjoint union of open balls and finitely many open annuli. A *semistable vertex set* of (X, D) is a semistable vertex set of X such that the points of D are contained in distinct open ball connected components of $X^{\text{an}} \setminus V$.

It is a consequence of the semistable reduction theorem of Deligne-Mumford that there exist semistable vertex sets of (X, D) : see [BPR11, Theorem 5.49]. In the sequel it will be convenient to consider a curve along with a choice of semistable vertex set, so we give such an object a name.

Definition 3.9. A *triangulated punctured curve* $(X, V \cup D)$ is a smooth, connected, proper algebraic K -curve X equipped with a finite set $D \subset X(K)$ of *punctures* and a semistable vertex set V of (X, D) .

Remark 3.10. This terminology is loosely based on that used in [Duc08] as well as the forthcoming book of Ducros on analytic spaces and analytic curves. Strictly speaking, what we have defined should be called a *semistably triangulated punctured curve*, but as these are the only triangulations that we consider, we will not need the added precision.

3.11. Let $(X, V \cup D)$ be a triangulated punctured curve, so

$$(3.11.1) \quad X^{\text{an}} \setminus (V \cup D) = A_1 \cup \cdots \cup A_n \cup \bigcup_{\alpha} B_{\alpha},$$

where each A_i is a generalized open annulus and $\{B_{\alpha}\}$ is an infinite collection of open balls. The *skeleton* of $(X, V \cup D)$ is the subset

$$\Sigma(X, V \cup D) = V \cup D \cup \Sigma(A_1) \cup \cdots \cup \Sigma(A_n).$$

(This set is denoted $\widehat{\Sigma}(X, V \cup D)$ in [BPR11].) For each i and each α the inclusions $\mathbf{H}(A_i) \hookrightarrow \mathbf{H}(X^{\text{an}})$ and $\mathbf{H}(B_{\alpha}) \hookrightarrow \mathbf{H}(X^{\text{an}})$ are local isometries [BPR11, Proposition 5.60]. For each open ball B_{α} the map $B_{\alpha} \hookrightarrow X^{\text{an}}$ extends to an inclusion $\bar{B}_{\alpha} \hookrightarrow X^{\text{an}}$ sending the end of B_{α} to a point $x_{\alpha} \in V$. We say that B_{α} is *adjacent* to x_{α} . For each open annulus A_i the map $A_i \hookrightarrow X^{\text{an}}$ extends to a continuous map $\bar{A}_i \hookrightarrow X^{\text{an}}$ sending the ends of A_i to points $x_i, y_i \in V$. We say that A_i is *adjacent* to x_i and y_i . The length $\ell(e_i) \in \Lambda$ of the geodesic segment $e_i = \Sigma(A_i) \cup \{x_i, y_i\}$ is then the modulus of A_i . We say that V is *strongly semistable* if $x_i \neq y_i$ for each open annulus A_i . For each generalized open annulus A_i the map $A_i \hookrightarrow X^{\text{an}}$ extends to a continuous map $\bar{A}_i \hookrightarrow X^{\text{an}}$ sending the end of A_i to a point $x_i \in V$ and

sending the puncture to a point $y_i \in D$. We say that A_i is *adjacent* to x_i and y_i . We define the length of $e_i = \Sigma(A_i) \cup \{x_i, y_i\}$ to be $\ell(e_i) = \infty$.

The skeleton $\Sigma = \Sigma(X, V \cup D)$ naturally has the structure of a Λ -metric graph with distinguished finite vertex set $V_f(\Sigma) = V$, infinite vertex set $V_\infty(\Sigma) = D$, and edges $\{e_1, \dots, e_n\}$ as above. Note that the Λ -points of Σ are exactly the type-2 points of X^{an} contained in Σ . For $x \in V$ the residue field $\widehat{\mathcal{H}}(x)$ of the completed residue field $\mathcal{H}(x)$ at the type-2 point x is a finitely generated field extension of k of transcendence degree 1; we let C_x be the smooth k -curve with function field $\mathcal{H}(x)$. For $x \in V$ we let $g(x)$ be the genus of C_x , and for $x \in \Sigma \setminus V$ we set $g(x) = 0$. These extra data give Σ the structure of an augmented Λ -metric graph. (In (3.22) we will see that Σ is in fact naturally a metrized complex of k -curves.) By the genus formula [BPR11, (5.45.1)], the genus of X is equal to the genus of the augmented Λ -metric graph Σ .

The open analytic domain $X^{\text{an}} \setminus \Sigma$ is isomorphic to an infinite disjoint union of open balls [BPR11, Lemma 5.18(3)]. If B is a connected component of $X^{\text{an}} \setminus \Sigma$, then the inclusion $B \hookrightarrow X^{\text{an}}$ extends to a map $\overline{B} \hookrightarrow X^{\text{an}}$ sending the end of B to a point of Σ . We define the *retraction* $\tau = \tau_\Sigma : X^{\text{an}} \rightarrow \Sigma$ by fixing Σ and sending a point $x \in X^{\text{an}}$ not in Σ to the end of the connected component of x in $X^{\text{an}} \setminus \Sigma$. This is a continuous map. If $x \in B_\alpha$ is in an open ball connected component of $X^{\text{an}} \setminus (V \cup D)$, then $\tau(x) \in V$ is the end of B_α , and if $x \in A_i$, then $\tau(x)$ coincides with the image of x under the retraction map $\tau : A_i \rightarrow \Sigma(A_i)$.

Here we collect some additional facts about skeleta from [BPR11, §5].

Proposition 3.12. *Let $(X, V \cup D)$ be a triangulated punctured curve with skeleton $\Sigma = \Sigma(X, V \cup D)$.*

- (1) *The skeleton Σ is the set of points in X^{an} that do not admit an open neighborhood isomorphic to $\mathbf{B}(1)_+$ and disjoint from $V \cup D$.*
- (2) *Let V_1 be a semistable vertex set of (X, D) such that $V_1 \supset V$. Then $\Sigma(X, V_1 \cup D) \supset \Sigma(X, V \cup D)$ and $\tau_{\Sigma(X, V \cup D)} = \tau_{\Sigma(X, V_1 \cup D)} \circ \tau_{\Sigma(X, V \cup D)}$.*
- (3) *Let $W \subset X^{\text{an}}$ be a finite set of type-2 points. Then there exists a semistable vertex set of (X, D) containing $V \cup W$.*
- (4) *Let $W \subset \Sigma$ be a finite set of type-2 points. Then $V \cup W$ is a semistable vertex set of (X, D) and $\Sigma(X, V \cup D) = \Sigma(X, V \cup W \cup D)$.*
- (5) *Let $x, y \in \Sigma \cap \mathbf{H}(X^{\text{an}})$. Then any geodesic segment from x to y in $\mathbf{H}(X^{\text{an}})$ is contained in Σ .*
- (6) *Let $x, y \in X^{\text{an}}$ be points of type-2 or 3 and let $[x, y]$ be a geodesic segment from x to y in $\mathbf{H}(X^{\text{an}})$. Then there exists a semistable vertex set V_1 of (X, D) such that $V_1 \supset V$ and $[x, y] \subset \Sigma(X, V_1 \cup D)$.*

Definition 3.13. A *skeleton* of (X, D) is a subset of X^{an} of the form $\Sigma = \Sigma(X, V \cup D)$ for some semistable vertex set V of (X, D) . Such a semistable vertex set V is called a *vertex set for Σ* . A *skeleton of X* is a skeleton of (X, \emptyset) .

The augmented Λ -metric graph structure of a skeleton Σ of (X, D) does not depend on the choice of vertex set for Σ .

3.14. Modifying the skeleton. Let X be a smooth, proper, connected K -curve and let $D \subset X(K)$ be a finite set. Let Σ be a skeleton of (X, D) , let $y \in X^{\text{an}} \setminus \Sigma$, let B be the connected component of $X^{\text{an}} \setminus \Sigma$ containing y , and let $x = \tau(y) \in \Sigma$ be the end of B . If $y \in \mathbf{H}(X^{\text{an}})$, then the geodesic segment $[x, y] \subset \overline{B}$ is the unique geodesic segment in $\mathbf{H}(X^{\text{an}})$ connecting x and y . If $y \in X(K)$, then we define $[x, y]$ to be the geodesic segment $[x, y] \subset \overline{B}$ as in (3.1). The following strengthening of Proposition 3.12(3) will be important in the sequel.

Lemma 3.15. *Let V be a semistable vertex set of (X, D) , let $\Sigma = \Sigma(X, V \cup D)$, let $W \subset X^{\text{an}}$ be a finite set of type-2 points, and let $E \subset X(K)$ be a finite set of type-1 points.*

- (1) *There exists a minimal semistable vertex set V_1 of $(X, D \cup E)$ which contains $V \cup W$, in the sense that any other such semistable vertex set contains V_1 .*

- (2) Let B be a connected component of $X^{\text{an}} \setminus \Sigma$ with end x and let y_1, \dots, y_n be the points of $(W \cup E) \cap B$. Then

$$\Sigma(X, V_1 \cup D \cup E) \cap \overline{B} = [x, y_1] \cup \dots \cup [x, y_n]$$

if $n > 0$, and $\Sigma(X, V_1 \cup D \cup E) \cap \overline{B} = \{x\}$ otherwise. Therefore

$$\Sigma(X, V_1 \cup D \cup E) = \Sigma \cup \bigcup_{y \in W \cup E} [\tau_\Sigma(y), y].$$

- (3) The skeleton $\Sigma_1 = \Sigma(X, V_1 \cup D \cup E)$ is minimal in the sense that any other skeleton containing Σ and $W \cup E$ must contain Σ_1 .

Proof. To prove the first part we may assume that $W = \{y\}$ consists of a single type-2 point not contained in Σ and $E = \emptyset$, or $E = \{y\}$ is a single type-1 point not contained in D and $W = \emptyset$. In the first case one sees using Lemma 3.4 that $V_1 = V \cup \{y, \tau(y)\}$ is the minimal semistable vertex set of (X, D) containing V and y , and in the second case $V_1 = V \cup \{\tau(y)\}$ is the minimal semistable vertex set of $(X, D \cup \{y\})$ containing V . For W and E arbitrary, let V_1 be the minimal semistable vertex set of $(X, D \cup E)$ containing $V \cup W$ and let $\Sigma_1 = \Sigma(X, V_1 \cup D)$. If $E = \emptyset$, then it is clear from Proposition 3.12(5) that $[x, y_i] \subset \Sigma_1$ for each i ; the other inclusion is proved by induction on n , adding one point at a time as above. The case $E \neq \emptyset$ is similar and is left to the reader. The final assertion follows easily from the first two and Proposition 3.12. \blacksquare

Remark 3.16. The above lemma shows in particular that the metric graph $\Sigma(X, V_1 \cup D \cup E)$ is obtained from $\Sigma(X, V \cup D)$ by a sequence of tropical modifications and their inverses (see Definition 2.28 and Example 2.32). It is easy to see that any tropical modification of $\Sigma(X, V \cup D)$ is of the form $\Sigma(X, V_1 \cup D \cup E)$ for a semistable vertex set V_1 which contains V and a finite subset $E \subset X(K)$.

3.17. The minimal skeleton. The Euler characteristic of $X \setminus D$ is defined to be $\chi(X \setminus D) = 2 - 2g(X) - \#D$, where $g(X)$ is the genus of X . We say that (X, D) is *stable* if $\chi(X \setminus D) < 0$. A semistable vertex set V of (X, D) is *minimal* if there is no semistable vertex set of (X, D) properly contained in V , and V is *stable* if there is no $x \in V$ of genus zero (resp. one) and valency less than 3 (resp. one) in $\Sigma(X, V \cup D)$. It is clear that minimal semistable vertex sets exist. A skeleton Σ of (X, D) is *minimal* if there is no skeleton of (X, D) properly contained in Σ .

The following consequence of the stable reduction theorem can be found in [BPR11, Theorem 5.49].

Proposition 3.18. Let V be a minimal semistable vertex set of (X, D) and let $\Sigma = \Sigma(X, V \cup D)$.

- (1) If $\chi(X \setminus D) \leq 0$, then Σ is the unique minimal skeleton of (X, D) ; moreover, Σ is equal to the set of points of $X^{\text{an}} \setminus D$ that do not admit an open neighborhood isomorphic to $\mathbf{B}(1)_+$ and disjoint from D .
- (2) If $\chi(X \setminus D) < 0$, then V is the unique minimal semistable vertex set of (X, D) , V is stable, and $V = \{x \in \Sigma : x \text{ has valency } \geq 3 \text{ or genus } \geq 1\}$.

Remark 3.19. We have $\chi(X \setminus D) > 0$ if and only if $X \cong \mathbf{P}^1$ and D contains at most one point; in this case, any type-2 point of X^{an} serves as a minimal semistable vertex set of (X, D) , and there does not exist a unique minimal skeleton. If $\chi(X \setminus D) = 0$, then either $X \cong \mathbf{P}^1$ and D consists of two points, or X is an elliptic curve and D is empty. In the first case, the minimal skeleton Σ of (X, D) is the extended line connecting the points of D and any type-2 point on Σ is a minimal semistable vertex set. If X is an elliptic curve and X^{an} contains a type-2 point x of genus 1, then $\{x\}$ is both the unique minimal semistable vertex set and the minimal skeleton. Otherwise X is a Tate curve, Σ is a circle, and any type-2 point of Σ is a minimal semistable vertex set. See [BPR11, Remark 5.51].

3.20. Tangent directions and the slope formula. As above we let X be a smooth, proper, connected algebraic K -curve. A continuous function $F : X^{\text{an}} \rightarrow \mathbf{R} \cup \{\pm\infty\}$ is called *piecewise affine* provided that $F(\mathbf{H}(X^{\text{an}})) \subset \mathbf{R}$ and $F \circ \alpha : [a, b] \rightarrow \mathbf{R}$ is a piecewise affine function for every geodesic segment $\alpha : [a, b] \hookrightarrow \mathbf{H}(X^{\text{an}})$. To any point $x \in X^{\text{an}}$ is associated a set T_x of *tangent directions* at x , defined as the set of germs of geodesic segments in X^{an} beginning at x . If $F : X^{\text{an}} \rightarrow \mathbf{R} \cup \{\pm\infty\}$ is a piecewise

affine function and $v \in T_x$ we denote by $d_v F(x)$ the outgoing slope of F in the direction v . We say that F is *harmonic* at a point $x \in X^{\text{an}}$ provided that there are only finitely many $v \in T_x$ with $d_v F(x) \neq 0$, and $\sum_{v \in T_x} d_v F(x) = 0$. See [BPR11, 5.65].

Let $x \in X^{\text{an}}$ be a type-2 point, let $\mathcal{H}(x)$ be the completed residue field of X^{an} at x , and let $\widetilde{\mathcal{H}}(x)$ be its residue field, as in (3.11). Then $\widetilde{\mathcal{H}}(x)$ is a finitely generated field extension of k of transcendence degree 1, and there is a canonical bijection between the set T_x of tangent directions to X^{an} at x and the set $\text{DV}(\widetilde{\mathcal{H}}(x)/k)$ of discrete valuations on $\widetilde{\mathcal{H}}(x)$ which are trivial on k . For $v \in T_x$ we let $\text{ord}_v : \widetilde{\mathcal{H}}(x) \rightarrow \mathbb{Z}$ denote the corresponding valuation. Let f be a nonzero analytic function on X^{an} defined on a neighborhood of x , let $c \in K^\times$ be such that $|c| = |f(x)|$, and let $\tilde{f}_x \in \widetilde{\mathcal{H}}(x)$ denote the residue of $c^{-1}f$. Then \tilde{f}_x is defined up to multiplication by k^\times , so for any $v \in T_x$ the integer $\text{ord}_v(\tilde{f}_x)$ is well-defined.

The following theorem is called the slope formula in [BPR11, Theorem 5.69]:

Theorem 3.21. *Let $f \in K(X)^\times$ be a nonzero rational function on X and let $F = -\log |f| : X^{\text{an}} \rightarrow \mathbb{R} \cup \{\pm\infty\}$. Let $D \subset X(K)$ contain the zeros and poles of f and let Σ be a skeleton of (X, D) . Then:*

- (1) $F = F \circ \tau$, where $\tau : X^{\text{an}} \rightarrow \Sigma$ is the retraction.
- (2) F is piecewise affine with integer slopes, and F is affine on each edge of Σ (with respect to a choice of vertex set V).
- (3) If x is a type-2 point of X^{an} and $v \in T_x$, then $d_v F(x) = \text{ord}_v(\tilde{f}_x)$.
- (4) F is harmonic at all points of $X^{\text{an}} \setminus D$.
- (5) Let $x \in X(K)$ and let v be the unique tangent direction at x . Then $d_v F(x) = \text{ord}_x(f)$.

3.22. The skeleton as a metrized complex of curves. Let $(X, V \cup D)$ be a triangulated punctured curve with skeleton $\Sigma = \Sigma(X, V \cup D)$. Recall that Σ is an augmented Λ -metric graph with infinite vertices D . We enrich Σ with the structure of a Λ -metrized complex of k -curves as follows. For $x \in V$ let C_x be the smooth, proper, connected k -curve with function field $\widetilde{\mathcal{H}}(x)$ as in (3.11). By definition C_x has genus $g(x)$. We have natural bijections

$$(3.22.1) \quad T_x \cong \text{DV}(\widetilde{\mathcal{H}}(x)/k) \cong C_x(k)$$

where the first bijection $v \mapsto \text{ord}_v$ is defined in (3.20) and the second associates to a closed point $\xi \in C_x(k)$ the discrete valuation ord_ξ on the function field $\widetilde{\mathcal{H}}(x)$ of C_x . Let $v \in T_x$ be a tangent direction and define $\text{red}_x(v)$ to be the point of C_x corresponding to the discrete valuation $\text{ord}_v \in \text{DV}(\widetilde{\mathcal{H}}(x)/k)$. These data make Σ into a Λ -metrized complex of k -curves.

3.23. Lifting metrized complexes of curves. We now prove that every metrized complex of curves over k arises as the skeleton of a smooth, proper, connected K -curve. This fact appears in the literature in various contexts (over discretely-valued fields): see for instance [Bak08, Appendix B] and [Sai96, Lemme 6.3]. For this reason we only sketch a proof using our methods.

Theorem 3.24. *Let \mathcal{C} be a Λ -metrized complex of k -curves. There exists a triangulated punctured curve $(X, V \cup D)$ such that the skeleton $\Sigma(X, V \cup D)$ is isomorphic to \mathcal{C} .*

Proof. Let C be a smooth, proper, connected k -curve. By elementary deformation theory, there is a smooth, proper admissible formal R -scheme \mathfrak{C} with special fiber C . By GAGA the analytic generic fiber \mathfrak{C}_K is the analytification of a smooth, proper, connected K -curve \mathcal{C} . There is a reduction map $\text{red} : \mathfrak{C}_K \rightarrow C$ from the analytic generic fiber of \mathfrak{C} to (the set underlying) C , under which the inverse image of the generic point of C is a single distinguished point x . The set $\{x\}$ is a semistable vertex set of \mathcal{C} , with associated skeleton also equal to $\{x\}$. Moreover, there is a canonical identification of $\widetilde{\mathcal{H}}(x)$ with the field of rational functions on \overline{C} by [Ber90, Proposition 2.4.4]. See (5.6) for a more detailed discussion of the relationship between semistable vertex sets and admissible formal models.

Let Γ be the metric graph underlying \mathcal{C} . Let V be the vertices of Γ and let E be its edges. By adding valence-2 vertices we may assume that Γ has no loop edges. Assume for the moment that Γ has no infinite edges. For a vertex $x \in V$ let C_x denote the smooth, proper, connected k -curve

associated to x , and choose an admissible formal curve \mathfrak{C}_x with special fiber isomorphic to C_x as above. For clarity we let $\text{red}_{\mathfrak{C}_x}$ denote the reduction map $(\mathfrak{C}_x)_K \rightarrow C_x$. By [BL93, Proposition 2.2], for every $\bar{x} \in C_x(k)$ the formal fiber $\text{red}_{\mathfrak{C}_x}^{-1}(\bar{x})$ is isomorphic to $\mathbf{B}(1)_+$. Let e be an edge of Γ with endpoints x, y , and let $\bar{x}_e \in C_x(k), \bar{y}_e \in C_y(k)$ be the reductions of the tangent vectors in the direction of e at x, y , respectively. Remove closed balls from $\text{red}_{\mathfrak{C}_x}^{-1}(\bar{x}_e)$ and $\text{red}_{\mathfrak{C}_y}^{-1}(\bar{y}_e)$ whose radii are such that the remaining open annuli have modulus equal to $\ell(e)$. We form a new analytic curve X^{an} by gluing these annuli together using some isomorphism of annuli for each edge e . The resulting curve is proper, hence is the analytification of an algebraic curve X . By construction the image of the set of distinguished points of the curves $(\mathfrak{C}_x)_K$ in X^{an} is a semistable vertex set, and the resulting skeleton (considered as a metric graph) is isomorphic to Γ .

If Γ does have infinite edges, then we apply the above procedure to the finite part of Γ , then puncture the curve X in the formal fibers over the smooth points of the residue curves which correspond to the directions of the infinite tails of Γ . ■

As an immediate application, we obtain the following property of the “abstract tropicalization map” from the moduli space of stable marked curves to the moduli space of stable abstract tropical curves:

Corollary 3.25. *If g and n are nonnegative integers with $2 - 2g - n < 0$, the natural map $\text{trop} : M_{g,n} \rightarrow M_{g,n}^{\text{trop}}$ (see [BPR11, Remark 5.52]) is surjective.*

See [ACP12] for an interpretation of the above map as a contraction from $M_{g,n}^{\text{an}}$ onto its skeleton (which also implies surjectivity).

4. MORPHISMS BETWEEN CURVES AND THEIR SKELETA

This section is a relative version of the previous one, in that we propose to study the behavior of semistable vertex sets and skeleta under finite morphisms of curves. We introduce finite morphisms of triangulated punctured curves and we prove that any finite morphism of punctured curves can be enriched to a finite morphism of triangulated punctured curves. This powerful result can be used to prove the simultaneous semistable reduction theorems of Coleman, Liu-Lorenzini, and Liu, which we do in Section 5. It is interesting to note that we use only analytic methods on analytic K -curves, making (almost) no explicit reference to semistable models; hence our proofs of the results of Liu-Lorenzini and Liu are very different from theirs. Using a relative version of the slope formula, we also show that a finite morphism of triangulated punctured curves induces (by restricting to skeleta) a finite harmonic morphism of metrized complexes of curves.

4.1. Morphisms between open balls and generalized open annuli. The main results of this section rest on a careful study of the behavior of certain morphisms between open balls and generalized open annuli. Some of the lemmas and intermediate results appear in some form in the literature — see in particular [Ber90] and [Ber93] — although they are hard to find in the form we need them. As the proofs are not difficult, for the convenience of the reader we include complete arguments.

Lemma 4.2. *Let $a, a' \in \mathfrak{m}_R$ and let $\varphi : \mathbf{S}(a')_+ \rightarrow \mathbf{S}(a)_+$ be a morphism.*

- (1) *If $\text{val} \circ \varphi$ is constant on $\Sigma(\mathbf{S}(a')_+)$, then $\varphi(\mathbf{S}(a')_+)$ is contained in $\mathbf{S}(a)_+ \setminus \Sigma(\mathbf{S}(a)_+)$.*
- (2) *If $\text{val} \circ \varphi$ is not constant on $\Sigma(\mathbf{S}(a')_+)$, then $\varphi(\Sigma(\mathbf{S}(a')_+)) \subset \Sigma(\mathbf{S}(a))$ and the restriction of φ to $\Sigma(\mathbf{S}(a')_+)$ has the form $x \mapsto m \cdot x + b$ for some nonzero integer m and some $b \in \Lambda$.*
- (3) *If φ takes an end of $\mathbf{S}(a')_+$ to an end of $\mathbf{S}(a)_+$, then $\text{val} \circ \varphi$ is not constant on $\Sigma(\mathbf{S}(a')_+)$.*

Proof. Part (2) is exactly [BPR11, Proposition 5.5], so suppose that $\text{val} \circ \varphi$ is constant on $\Sigma(\mathbf{S}(a))$. The map $\varphi : \mathbf{S}(a')_+ \rightarrow \mathbf{S}(a)_+ \subset \mathbf{G}_m^{\text{an}}$ is given by a unit f on $\mathbf{S}(a')_+$. By [BPR11, Proposition 5.2], we can write $f = \alpha(1 + g)$, where $\alpha \in K^\times$ and $|g(x')| < 1$ for all $x' \in \mathbf{S}(a')_+$. Hence $|\varphi(x') - \alpha| < |\alpha|$ for all $x' \in \mathbf{S}(a')_+$; here $|\varphi(x') - \alpha|$ should be interpreted as the absolute value of the function $f - \alpha$ in $\mathcal{H}(x')$, or equivalently as the absolute value of the function $t - \alpha$ in $\mathcal{H}(\varphi(x))$, where t is a parameter on \mathbf{G}_m . But $|t(x) - \alpha| \geq |\alpha|$ for every $x \in \Sigma(\mathbf{S}(a))$, so $\varphi(\mathbf{S}(a')_+) \cap \Sigma(\mathbf{S}(a)) = \emptyset$.

Let x' be an end of $\mathbf{S}(a')_+$, let x be an end of $\mathbf{S}(a)_+$, and suppose that $\varphi(x') = x$. Since x' is a limit point of $\Sigma(\mathbf{S}(a')_+)$, there exists a sequence of points $x'_1, x'_2, \dots \in \Sigma(\mathbf{S}(a')_+)$ converging to x' . Then $\varphi(x'_1), \varphi(x'_2), \dots$ converge to x , so $\text{val}(\varphi(x'_1)), \text{val}(\varphi(x'_2)), \dots$ converges to $\text{val}(x)$. Since each $\text{val}(\varphi(x'_i)) \in (0, \text{val}(a))$ but $\text{val}(x) \notin (0, \text{val}(a))$, $\text{val} \circ \varphi$ is not constant on $\Sigma(\mathbf{S}(a')_+)$. ■

Lemma 4.3. *Let B' be an open ball, let A be a generalized open annulus, and let $\varphi : B' \rightarrow A$ be a morphism. Then $\varphi(B') \cap \Sigma(A) = \emptyset$, and φ extends to a continuous map $\overline{B'} \rightarrow A$.*

Proof. The assertions of the Lemma are clear if φ is constant, so assume that φ is non-constant. We identify A with $\mathbf{S}(a)_+$ for $a \in \mathfrak{m}_R$ and B' with $\mathbf{B}(1)_+$. Any unit on B' has constant absolute value, so $\text{val} \circ \varphi$ is constant on B' . Since $\mathbf{B}(1)_+ \setminus \{0\}$ is a generalized open annulus, and since the type-1 point 0 maps to a type-1 point of A , it follows from Lemma 4.2 that $\varphi(B') \cap \Sigma(A) = \emptyset$. Let B be the connected component of $A \setminus \Sigma(A)$ containing $\varphi(B')$. Then the morphism $B' \rightarrow B \hookrightarrow A$ extends to a continuous map $\overline{B'} \rightarrow \overline{B} \hookrightarrow A$. ■

Definition 4.4. Let T be a metric space and let $m \in \mathbf{R}_{>0}$. A continuous injection $\varphi : [a, b] \hookrightarrow T$ is an *embedding with expansion factor m* provided that $r \mapsto \varphi(r/m) : [ma, mb] \hookrightarrow T$ is a geodesic segment. A continuous injection $\varphi : [a, b] \hookrightarrow T$ is *piecewise affine* provided that there exist $a = a_0 < a_1 < \dots < a_r = b$ and $m_1, \dots, m_r \in \mathbf{R}_{>0}$ such that $\varphi|_{[a_{i-1}, a_i]}$ is an embedding with expansion factor m_i for each $i = 1, \dots, r$.

Lemma 4.5. *Let B, B' be open balls, let x' be the end of B , and let $\varphi : B' \rightarrow B$ be a morphism with finite fibers. Let $y' \in B'$, let $y = \varphi(y')$, and let $x = \varphi(x')$. Then the restriction of φ to the geodesic segment $[x', y']$ is injective, and $\varphi([x', y'])$ is equal to the geodesic segment $[x, y]$. If in addition there exists $N > 0$ such that all fibers of φ have fewer than N elements, then $\varphi|_{[x', y']}$ is piecewise affine.*

Proof. First suppose that y' does not have type 4. Choose isomorphisms $B' \cong \mathbf{B}(1)_+$ such that $0 > y'$ and $B \cong \mathbf{B}(1)_+$ such that $\varphi(0) = 0$. Since $y' \in [x', 0]$ we may replace $[x', y']$ by the larger geodesic segment $[x', 0]$ to assume that $y' = 0$. Define $F : \Sigma(\mathbf{S}(0)_+) \rightarrow \mathbf{R}$ by $F(x) = \text{val} \circ \varphi(x) = -\log |\varphi(x)|$. By [BPR11, Proposition 5.10], F is a piecewise affine function, and for any point $z' \in \Sigma(\mathbf{S}(0)_+)$ at which F is differentiable, the derivative of F is equal to the number of zeros q' of φ with $\text{val}(q') > \text{val}(z')$. It follows that F is monotonically increasing, so φ is injective on $[x', 0]$. Let

$$Z = \{\tau(q') : q' \in \mathbf{S}(0)_+, \varphi(q') = 0\} \subset \Sigma(\mathbf{S}(0)_+).$$

If $C \subset \Sigma(\mathbf{S}(0)_+) \setminus Z$ is a connected component, then $A' = \tau^{-1}(C)$ is a generalized open annulus mapping to $\mathbf{S}(0)_+$. By the above, $\text{val} \circ \varphi$ is not constant on $\Sigma(A') = \Sigma(\mathbf{S}(0)_+) \cap A'$, so by Lemma 4.2, $\varphi(\Sigma(A')) \subset \Sigma(\mathbf{S}(0)_+)$. Since $\Sigma(\mathbf{S}(0)_+) \setminus Z$ is dense in $\Sigma(\mathbf{S}(0)_+)$, we have that $[x', 0] = \Sigma(\mathbf{S}(0)_+) \cup \{0\}$ maps bijectively onto $[x, 0]$. It is clear in this case that the restriction of φ to the closure of any connected component of $\Sigma(\mathbf{S}(0)_+) \setminus Z$ is an embedding with integer expansion factor, so $\varphi|_{[x', 0]}$ is piecewise affine. Note that $\varphi|_{[x', 0]}$ changes expansion factor at most $\#\varphi^{-1}(0)$ times.

Now suppose that y' has type 4. Suppose that $z', w' \in [x', y']$ are two distinct points such that $\varphi(z') = \varphi(w')$. Assume without loss of generality that $z' < w'$. Since w' and z' have the same image under φ , they both have the same type, so $w' \neq y'$ because y' is the only type-4 point in $[x', y']$. Applying the above to the geodesic $[x', w']$ gives a contradiction. Therefore, φ is injective on $[x', y']$, so $\varphi([x', y']) = [x, y]$ since B is uniquely path-connected ([BR10, Corollary 1.14]).

Assume now that all fibers of φ have size bounded by N . The above argument proves that for all $z' < y'$, the restriction of φ to $[x', z']$ is piecewise affine, and $\varphi|_{[x', z']}$ changes expansion factor at most N times. Therefore there exists $z'_0 < y'$ and $m \in \mathbf{Z}_{>0}$ such that for all $z' \in [z'_0, y']$, the restriction of φ to $[z'_0, z']$ is an embedding with expansion factor m . It follows that $\varphi|_{[z'_0, y']}$ is an embedding with expansion factor m , so $\varphi|_{[x', y']}$ is piecewise affine. ■

Lemma 4.6. *Let B, B' be open balls and let $\varphi : B' \rightarrow B$ be a morphism. Suppose that φ is open, separated, and has finite fibers. Let x' be the end of B' and let $x = \varphi(x') \in \overline{B}$. Then $B_1 = \varphi(B')$ is an open ball connected component of $\overline{B} \setminus \{x\}$ and $\varphi : B' \rightarrow B_1$ is finite and order-preserving. In particular, if x is the end of B , then $\varphi : B' \rightarrow B$ is finite and order-preserving.*

Proof. Let $y' \in B'$. By Lemma 4.5, the restriction of φ to the geodesic $[x', y']$ is injective, so $\varphi(y') \neq x$. Therefore $x \notin B_1$. Let C be the connected component of $B \setminus \{x\}$ containing B_1 . Since x is an end of C and x' maps to x , Lemma 4.3 implies that C is not an open annulus, so it is an open ball. By hypothesis, B_1 is an open subset of C ; since $B' \cup \{x'\}$ is compact, $\varphi(B' \cup \{x'\}) = B_1 \cup \{x\}$ is closed in \overline{B} , and therefore $B_1 = (B_1 \cup \{x\}) \cap C$ is closed in C . Since C is connected, we have $B_1 = C$.

Since φ has finite fibers, $\varphi : B' \rightarrow B_1$ is finite if and only if it is proper by [Ber90, Corollary 3.3.8]. If $D \subset B_1$ is compact, then $\varphi^{-1}(D)$ is compact since $\varphi^{-1}(D)$ is closed as a subset of the compact space $\overline{B'}$. One has $\text{Int}(B'/B_1) = B'$ by [Ber90, Proposition 3.1.3(i)] since B' is a boundaryless K -analytic space. Therefore φ is proper.

The fact that $\varphi : B' \rightarrow B_1$ is order-preserving follows immediately from Lemma 4.5. \blacksquare

4.7. Let $\varphi : B' \rightarrow B$ be a finite morphism of open balls. Since an open ball is a smooth curve, φ is flat in the sense that if $\mathcal{M}(A) \subset B$ is an affinoid domain and $\varphi^{-1}(\mathcal{M}(A)) = \mathcal{M}(A')$, then A' is a (finite) flat A -algebra. For $x \in \mathcal{M}(A)$ the fiber over x is

$$\varphi^{-1}(x) = \mathcal{M}(A' \widehat{\otimes}_A \mathcal{H}(x)) = \mathcal{M}(A' \otimes_A \mathcal{H}(x)),$$

where the second equality holds because A' is a finite A -algebra. It follows that for $x \in B$ the quantity $\sum_{x' \mapsto x} \dim_{\mathcal{H}(x)} \mathcal{O}_{\varphi^{-1}(x), x'}$ is independent of x ; we call this number the *degree* of φ .

Proposition 4.8. *Let B, B' be open balls and let $\varphi : B' \rightarrow B$ be a finite morphism. Let x' be the end of B' , let $x = \varphi(x')$ be the end of B , let $y \in B$, and let $y'_1, \dots, y'_n \in B'$ be the inverse images of y in B' . Then*

$$\varphi^{-1}([x, y]) = [x', y'_1] \cup \dots \cup [x', y'_n].$$

Proof. Let $T' = [x', y'_1] \cup \dots \cup [x', y'_n]$. The inclusion $T' \subset \varphi^{-1}([x, y])$ follows from Lemma 4.5. First we claim that for $z \in [x, y]$ near enough to x , there is only one preimage of z in B' . Shrinking $[x, y]$ if necessary, we may assume that y has type 2. Choose a type-1 point $w \in B$ such that $w > y$, so $(x, y] \subset (x, w) = \Sigma(B \setminus \{w\})$. Let $w' \in B'$ be a preimage of w and choose $z' \in \Sigma(B' \setminus \{w'\})$ such that $z' \leq \tau(q')$ for all $q' \in \varphi^{-1}(w)$, where $\tau : B' \setminus \{w'\} \rightarrow \Sigma(B' \setminus \{w'\})$ is the retraction. Let A' be the open annulus $\tau^{-1}((x', z')) \subset B'$. Then $\varphi(A') \subset B \setminus \{w\}$, and φ takes the end x' of A' to the end x of $B \setminus \{w\}$, so by Lemma 4.2(2,3), $\varphi(\Sigma(A')) \subset \Sigma(B \setminus \{w\})$ and the map $\Sigma(A') \rightarrow \Sigma(B \setminus \{w\})$ is injective. If $u' \in A' \setminus \Sigma(A')$, then u' is contained in an open ball in A' , so $\varphi(u') \notin \Sigma(B \setminus \{w\})$ by Lemma 4.3. Therefore every point of $\Sigma(B \setminus \{w\})$ has at most one preimage in A' . Let $u' \in B' \setminus A'$, so $u' \geq z'$. Then $\varphi(u') \geq \varphi(z')$, so every point $u \in (x, y]$ with $u < \varphi(z')$ has exactly one preimage in B' (note that $\varphi(z') \in \Sigma(B \setminus \{w\})$ by Lemma 4.5).

Let d be the degree of the finite morphism φ , so for every $z \in B$ we have

$$d = \sum_{z' \in \varphi^{-1}(z)} \dim_{\mathcal{H}(z)} \mathcal{O}_{\varphi^{-1}(z), z'}.$$

For $y \in B$ as in the statement of the Proposition, define a function $\delta : (x, y] \rightarrow \mathbf{Z}$ by

$$\delta(z) = \sum_{\substack{z' \in \varphi^{-1}(z) \\ z' \in T'}} \dim_{\mathcal{H}(z)} \mathcal{O}_{\varphi^{-1}(z), z'}.$$

Clearly $\delta(z) \leq d$ for all $z \in (x, y]$, and $\delta(z) = d$ if and only if $\varphi^{-1}(z) \subset T'$. By definition of T' we have $\delta(y) = d$, and by the above, $\delta(z) = d$ for $z \in (x, y]$ close enough to x (any geodesic segment $[x', y'_i]$ surjects onto $[x, y]$, hence contains the unique preimage of z). Therefore it is enough to show that $\delta(z_1) \geq \delta(z_2)$ if $z_1 \leq z_2$.

If $z \in B$ is a point of type 2 or 3, then $B_z := \{w \in B : w \geq z\}$ is a (not necessarily strict) affinoid subdomain of B . If $w' \in B'$ is such that $\varphi(w') \geq z$, then $\varphi([x', w']) = [x, \varphi(w')]$ is a geodesic containing z , so there exists $z' \in [x', w']$ mapping to z . It follows that

$$\varphi^{-1}(B_z) = \coprod_{z' \mapsto z} \{w' \in B' : w' \geq z'\} = \coprod_{z' \mapsto z} B_{z'}$$

is a disjoint union of affinoid domains. Hence each map $B_{z'} \rightarrow B_z$ is finite, and its degree is equal to $\dim_{\mathcal{H}(z)} \mathcal{O}_{\varphi^{-1}(z), z'}$. Let $z_1, z_2 \in (x, y]$ and assume that $z_1 < z_2$, so z_1 has type 2 or 3. For $z'_1 \in \varphi^{-1}(z_1)$ we have

$$\dim_{\mathcal{H}(z_1)} \mathcal{O}_{\varphi^{-1}(z_1), z'_1} = \sum_{\substack{z'_2 \mapsto z_2 \\ z'_2 \geq z'_1}} \dim_{\mathcal{H}(z_2)} \mathcal{O}_{\varphi^{-1}(z_2), z'_2}.$$

Summing over all $z'_1 \in \varphi^{-1}(z_1) \cap T'$, we obtain

$$\begin{aligned} \delta(z_1) &= \sum_{\substack{z'_1 \mapsto z_1 \\ z'_1 \in T'}} \dim_{\mathcal{H}(z_1)} \mathcal{O}_{\varphi^{-1}(z_1), z'_1} = \sum_{\substack{z'_1 \mapsto z_1 \\ z'_1 \in T'}} \sum_{\substack{z'_2 \mapsto z_2 \\ z'_2 \geq z'_1}} \dim_{\mathcal{H}(z_2)} \mathcal{O}_{\varphi^{-1}(z_2), z'_2} \\ &\geq \sum_{\substack{z'_1 \mapsto z_1 \\ z'_1 \in T'}} \sum_{\substack{z'_2 \mapsto z_2 \\ z'_2 \geq z'_1 \\ z'_2 \in T'}} \dim_{\mathcal{H}(z_2)} \mathcal{O}_{\varphi^{-1}(z_2), z'_2} = \delta(z_2), \end{aligned}$$

where the final equality holds because if $z'_2 \in \varphi^{-1}(z_2) \cap T'$, then there exists $z'_1 \in \varphi^{-1}(z_1) \cap T'$ such that $z'_2 \geq z'_1$, namely, the unique point of $[x', z'_2]$ mapping to z_1 . \blacksquare

Proposition 4.9. *Let $a \in \mathfrak{m}_R \setminus \{0\}$, let $A' = \mathbf{S}(a)_+$, let B be an open ball, and let $\varphi : A' \rightarrow B$ be a morphism with finite fibers. Suppose that each end of A' maps to the end of B or to a type-2 point of B under the induced map $\overline{A'} \rightarrow \overline{B}$. Let $\alpha = \varphi \circ \sigma : [0, \text{val}(a)] \rightarrow \overline{B}$. Then there exist finitely many numbers $r_0, r_1, r_2, \dots, r_n \in \Lambda$ with $0 = r_0 < r_1 < r_2 < \dots < r_n = \text{val}(a)$ such that α is an embedding with nonzero integer expansion factor when restricted to each interval $[r_i, r_{i+1}]$. In other words, α is piecewise affine with integer expansion factors. Moreover, the image of α is a geodesic segment between type-2 points of B .*

Proof. Let x' be the end $\sigma(0)$ of A' and let $x = \varphi(x')$. Suppose first that x is the end of B . Choose an identification $B \cong \mathbf{B}(1)_+$, and let $r_+ = \min\{\text{val}(y') : \varphi(y') = 0\} > 0$. Let A'_+ be the open annulus $\text{val}^{-1}((0, r_+)) \subset A'$. Then $\varphi(A'_+) \subset \mathbf{B}(1)_+ \setminus \{0\} = \mathbf{S}(0)_+$ and the end x' of A'_+ maps to the end x of $\mathbf{S}(0)_+$, so by Lemma 4.2(2,3), α is an embedding with (nonzero) integer expansion factor when restricted to $[0, r_+]$.

Now suppose that x is not the end of B . Let $r = \min\{\text{val}(y') : \varphi(y') = x\} > 0$ and let $A'' = \text{val}^{-1}((0, r))$. Then $\varphi(A'')$ is contained in a connected component C of $B \setminus \{x\}$, and φ takes the end x' of A'' to the end x of C . If C is an open annulus, then α is an embedding with integer expansion factor when restricted to $[0, r]$ by Lemma 4.2(2,3), and if C is an open ball, then we proceed as above to find $r_+ \in (0, r) \cap \Lambda$ such that α is an embedding with integer expansion factor when restricted to $[0, r_+]$.

Applying the above argument to the morphism φ composed with the automorphism $t \mapsto a/t$ of $\mathbf{S}(a)_+$ (which interchanges the two ends), we find that there exists $r_- \in [0, \text{val}(a)) \cap \Lambda$ such that α is an embedding with integer expansion factor when restricted to $[r_-, \text{val}(a)]$. Let $s \in (0, \text{val}(a)) \cap \Lambda$. Replacing A' with the annulus $\text{val}^{-1}((0, s))$ (resp. $\text{val}^{-1}((s, \text{val}(a)))$), the above arguments then provide us with $s_+ \in (s, \text{val}(a)) \cap \Lambda$ (resp. $s_- \in [0, s) \cap \Lambda$) such that α is an embedding with integer expansion factor when restricted to $[s, s_+]$ (resp. $[s_-, s]$). The first assertions now follow because there is a finite subcover of the open covering

$$\{[0, r_+]\} \cup \{(r_-, \text{val}(a)]\} \cup \{(s_-, s_+) : s \in (0, r) \cap \Lambda\}$$

of the compact space $[0, r]$.

As for the final assertion, choose $0 = r_0 < r_1 < r_2 < \dots < r_n = \text{val}(a)$ such that α is an embedding with integer expansion factor on each $[r_i, r_{i+1}]$. Let $i_0 \in \{0, 1, \dots, n\}$ be the largest integer such that $\alpha(r_{i_0}) > \alpha(r_i)$ (in the canonical partial ordering on B) for all $i < i_0$. If $i_0 = n$, then we are done, so assume that $i_0 < n$. Let $y = \alpha(r_{i_0}) \in B$, and choose an identification $B \cong \mathbf{B}(1)_+$ such that $0 > y$. If $F = -\log |\varphi|$, then by [BPR11, Proposition 5.10], at every point $r \in (0, \text{val}(a))$ the change in slope of F at r is equal to the negative of the number of zeros of φ with valuation r (cf. the proof

of Lemma 4.5); in particular, the slope of F can only *decrease*. By construction F is monotonically increasing on $[0, r_{i_0}]$. Since $\alpha([r_{i_0}, r_{i_0+1}])$ is a geodesic segment, it meets y only at $\alpha(r_{i_0})$. The image of $(r_{i_0}, r_{i_0+1}]$ under α is not contained in an open ball connected component of $B \setminus \{y\}$ because $\alpha(r_{i_0+1}) \not\asymp y$; hence F is *decreasing* on an interval $[r_{i_0}, r_{i_0} + \varepsilon]$ for some $\varepsilon > 0$. Since the slope of F can only decrease, it follows that F is monotonically decreasing on $[r_{i_0}, r_n]$. It follows immediately from this that $\alpha([0, \text{val}(a)]) = [x, y]$ or $\alpha([0, \text{val}(a)]) = [\alpha(\text{val}(a)), y]$, whichever segment is larger. ■

4.10. Morphisms between curves and skeleta. In what follows we fix smooth, connected, proper algebraic K -curves X, X' and a finite morphism $\varphi : X' \rightarrow X$. Let $D \subset X(K)$ and $D' \subset X'(K)$ be finite sets of closed points. The map on analytifications $\varphi : X'^{\text{an}} \rightarrow X^{\text{an}}$ is finite and open by [Ber90, Lemma 3.2.4].

Proposition 4.11. *Let Σ' be a skeleton of (X', D') and let Σ be a skeleton of (X, D) . There exists a skeleton Σ_1 of $(X, D \cup \varphi(D'))$ containing $\Sigma \cup \varphi(\Sigma')$, and there is a minimal such Σ_1 with respect to inclusion.*

Proof. First we will prove the Proposition in the case $D = D' = \emptyset$. Let V be a vertex set for Σ and let V' be a vertex set for Σ' containing $\{\tau(y') : y' \in \varphi^{-1}(V)\}$. Let A' be an open annulus connected component of $X'^{\text{an}} \setminus V'$ and let $e' \subset \Sigma'$ be the associated edge. We claim that $\varphi(e')$ is a geodesic segment between type-2 points of X^{an} . Let C be the connected component of $X^{\text{an}} \setminus V$ containing $\varphi(A')$.

- (i) If C is an open ball then the claim follows immediately from Proposition 4.9.
- (ii) If C is an open annulus and $\varphi(A') \cap \Sigma(C) = \emptyset$, then A' is contained in an open ball in C because each connected component of $C \setminus \Sigma(C)$ is an open ball, so the claim is true as in (i).
- (iii) If C is an open annulus and $\varphi(A') \cap \Sigma(C) \neq \emptyset$, then $\varphi(e')$ is a geodesic segment in X^{an} by Lemma 4.2.

Applying the above to each edge e' of Σ' , we find that there exists a finite set of type-2 points $x_1, y_1, x_2, y_2, \dots, x_n, y_n \in X^{\text{an}}$ such that $\varphi(\Sigma') = \bigcup_{i=1}^n [x_i, y_i]$, where $[x_i, y_i]$ denotes a geodesic segment from x_i to y_i . Let

$$\Sigma_1 = \Sigma \cup \bigcup_{i=1}^n ([x_i, \tau_\Sigma(x_i)] \cup [y_i, \tau_\Sigma(y_i)]).$$

By Lemma 3.15(2) as applied to $W = \{x_1, y_1, \dots, x_n, y_n\}$, we have that Σ_1 is a skeleton of X , so by Proposition 3.12(5), Σ_1 contains $\Sigma \cup \varphi(\Sigma')$ since Σ_1 contains each geodesic segment $[x_i, y_i]$. Any other skeleton of X containing Σ and all x_i, y_i also contains the geodesics $[x_i, \tau_\Sigma(x_i)]$ and $[y_i, \tau_\Sigma(y_i)]$, so Σ_1 is the minimal such skeleton.

Now assume that $D \amalg D' \neq \emptyset$. By Lemma 3.15(3) there is a minimal skeleton Σ_1 of X containing Σ and $\varphi(D')$; replacing Σ by Σ_1 and D by $D \cup \varphi(D')$, we may assume without loss of generality that $\varphi(D') \subset D$. We will proceed by induction on the size of $D \amalg D'$. As above we let V be a vertex set for Σ and we let V' be a vertex set for Σ' containing $\{\tau(y') : y' \in \varphi^{-1}(V \cup D)\}$.

- (i) Suppose that D' is not empty. Let $x' \in D'$ and let Σ_1 be the minimal skeleton of (X, D) containing $\Sigma \cup \varphi(\Sigma(X', V' \cup D' \setminus \{x'\}))$. Let A' be the connected component of $X'^{\text{an}} \setminus (V' \cup D')$ whose closure contains x' , let $x = \varphi(x') \in D$, and let A be the connected component of $X^{\text{an}} \setminus (V \cup D)$ whose closure contains x . Note that $\Sigma' = \Sigma(X', V' \cup D' \setminus \{x'\}) \cup \Sigma(A') \cup \{x'\}$. Clearly $\varphi(A') \subset A$, and $\varphi^{-1}(x) \cap A' = \emptyset$ by construction. Since φ takes the puncture of A' to the puncture of A , one shows as in the proof of Lemma 4.2 that $\varphi(\Sigma(A')) \subset \Sigma(A)$. Therefore Σ_1 contains $\Sigma \cup \varphi(\Sigma')$.
- (ii) Now suppose that $D' = \emptyset$ and $D \neq \emptyset$. Let $x \in D$ and let Σ_1 be the minimal skeleton of $(X, D \setminus \{x\})$ containing $\Sigma(X, V \cup D \setminus \{x\}) \cup \varphi(\Sigma')$. Let $\Sigma_2 = \Sigma_1 \cup [x, \tau_\Sigma(x)]$. It follows from Lemma 3.15(3) that Σ_2 is the minimal skeleton of (X, D) containing Σ_1 . ■

Let V be a semistable vertex set of (X, D) . Recall from (3.11) that a connected component C of $X^{\text{an}} \setminus (V \cup D)$ is *adjacent* to a vertex $x \in V$ provided that the closure of C in X^{an} contains x . Proposition 4.12(1) below is exactly [Ber90, Theorem 4.5.3] when $D = D' = \emptyset$.

Proposition 4.12. *Suppose that $D' = \varphi^{-1}(D)$ and that one of the following two conditions holds:*

- (1) $\chi(X \setminus D) \leq 0$ (hence also $\chi(X' \setminus D') \leq 0$), or
- (2) $\varphi^{-1}(V) \subset V'$.

In the situation of (1) let Σ (resp. Σ') be the minimal skeleton of (X, D) (resp. (X', D')), and in (2) let $\Sigma = \Sigma(X, V \cup D)$ and $\Sigma' = \Sigma(X', V' \cup D')$. Then $\varphi^{-1}(\Sigma) \subset \Sigma'$.

Proof. First suppose that (2) holds. Let $x' \in X'^{\text{an}} \setminus \Sigma'$ and let B' be the connected component of $X'^{\text{an}} \setminus \Sigma'$ containing x' , so B' is an open ball. By hypothesis, $\varphi(B')$ is contained in a connected component C of $X^{\text{an}} \setminus (V \cup D)$. If C is an open ball, then $\varphi(B') \cap \Sigma = \emptyset$ by the definition of $\Sigma(X, V \cup D)$. If C is a generalized open annulus, then $\varphi(B') \cap \Sigma = \varphi(B') \cap \Sigma(C) = \emptyset$ by Lemma 4.3. Therefore $\varphi(x') \notin \Sigma$.

Now suppose that (1) holds. Let V be a semistable vertex set of (X, D) such that $\Sigma = \Sigma(X, V \cup D)$. By subdividing edges of Σ and enlarging V if necessary, we may and do assume that Σ has no loop edges. By Proposition 3.18(2), no point in V of genus zero has valence one in Σ . First we claim that if $B' \subset X'^{\text{an}} \setminus D'$ is an open analytic domain which is isomorphic to $\mathbf{B}(1)_+$, then $\varphi(B') \cap V = \emptyset$. If $\varphi^{-1}(V) \cap B'$ contains more than one point, then it is easy to see that there exists a smaller open ball $B'' \subset B'$ such that $\varphi^{-1}(V) \cap B''$ contains exactly one point. Replacing B' by B'' , we may assume that there is a unique point $y' \in \varphi^{-1}(V) \cap B'$. Let $y = \varphi(y') \in V$. Since $g(y') = 0$ we have $g(y) = 0$.

The open analytic domain $B' \setminus \{y'\}$ is the disjoint union of an open annulus A' and an infinite collection of open balls by Lemma 3.4. Each connected component C' of $B' \setminus \{y'\}$ maps into a connected component C of $X^{\text{an}} \setminus (V \cup D)$ adjacent to y , with the end y' of C' mapping to the end y of C . By Lemma 4.3, no open ball connected component of $B' \setminus \{y'\}$ can map to a generalized open annulus connected component of $X^{\text{an}} \setminus (V \cup D)$. There are at least two generalized open annulus connected components A of $X^{\text{an}} \setminus (V \cup D)$ adjacent to y , so some such A must satisfy $\varphi(B') \cap A = \emptyset$. But the map $\varphi : X'^{\text{an}} \rightarrow X^{\text{an}}$ is *open* by [Ber90, Lemma 3.2.4], so $\varphi(B')$ is an open neighborhood of y , which contradicts the fact that y is a limit point of A . This proves the claim.

Let V' be a semistable vertex set of (X', D') such that $\Sigma' = \Sigma(X', V' \cup D')$. Since $X' \setminus \Sigma'$ is a disjoint union of open balls, by the above we have $\varphi^{-1}(V) \subset \Sigma'$. Hence we may enlarge V' to contain $\varphi^{-1}(V)$ without changing Σ' , so we are reduced to (2). \blacksquare

Theorem 4.13. *Let Σ' be a skeleton of (X', D') and let Σ be a skeleton of (X, D) . Suppose that $\varphi(\Sigma') \subset \Sigma$, and if $X \cong \mathbf{P}^1$ assume in addition that there exists a type-2 point $z \in \Sigma$ such that $\varphi^{-1}(z) \subset \Sigma'$. Then $\varphi^{-1}(\Sigma)$ is a skeleton of $(X', \varphi^{-1}(D))$.*

We will require the following lemmas. Lemma 4.14 is similar to [Ber90, Corollary 4.5.4].

Lemma 4.14. *Let $B' \subset X'^{\text{an}}$ be an open analytic domain isomorphic to $\mathbf{B}(1)_+$ and let $B = \varphi(B') \subset X^{\text{an}}$. Then B is an open analytic domain of X^{an} , and one of the following is true:*

- (1) B is an open ball and $\varphi : B' \rightarrow B$ is finite and order-preserving.
- (2) $X \cong \mathbf{P}^1$ and $B = X^{\text{an}}$.

Proof. Suppose that the genus of X is at least one, so X' also has genus at least one. By Lemma 4.6 we only need to show that B is contained in an open ball in X^{an} . Let Σ (resp. Σ') be the minimal skeleton of X (resp. X'). By Proposition 3.18(1) we have $B' \subset X'^{\text{an}} \setminus \Sigma'$, so by Proposition 4.12(1) we have $B \subset X^{\text{an}} \setminus \Sigma$. But every connected component of $X^{\text{an}} \setminus \Sigma$ is an open ball, so B is contained in an open ball.

In the case $X = \mathbf{P}^1$, suppose first that $B'(K) \rightarrow \mathbf{P}^1(K)$ is not surjective, so we may assume that $\infty \notin B$ after choosing a suitable coordinate on \mathbf{P}^1 . Let x' be the end of B' and let $x = \varphi(x')$. Then x is a type-2 point, so $\varphi(B' \cup \{x'\}) = B \cup \{x\}$ is a compact subset of $\mathbf{A}^{1, \text{an}}$. Since $\mathbf{A}^{1, \text{an}}$ is covered by an increasing union of open balls, B is contained in an open ball.

Now suppose that $B'(K) \rightarrow \mathbf{P}^1(K)$ is surjective. Since $\mathbf{P}^1(K)$ is dense in $\mathbf{P}^{1,\text{an}}$ and $B \cup \{x\}$ is closed in $\mathbf{P}^{1,\text{an}}$, it follows that $B \cup \{x\} = \mathbf{P}^{1,\text{an}}$. If $x \notin B$, then B is contained in a connected component of $\mathbf{P}^{1,\text{an}} \setminus \{x\}$, which contradicts the surjectivity of $B'(K) \rightarrow \mathbf{P}^1(K)$. Therefore $B = \mathbf{P}^{1,\text{an}}$. ■

Remark 4.15. Case (2) of Lemma 4.14 does occur. For instance, let $\varphi : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ be the finite morphism $t \mapsto t^2$, and let B' be the open ball in $\mathbf{P}^{1,\text{an}}$ obtained by deleting the closed ball of radius $1/2$ around $1 \in \mathbf{P}^1(K)$. If $\text{char}(k) \neq 2$, then for every point $x \in \mathbf{P}^1(K)$, either $x \in B'(K)$ or $-x \in B'(K)$, so $B' \rightarrow \mathbf{P}^{1,\text{an}}$ is surjective.

Lemma 4.16. *Let V be a semistable vertex set of (X, D) , let $\Sigma = \Sigma(X, V \cup D)$, and let $B \subset X$ be an open analytic domain isomorphic to $\mathbf{B}(1)_+$ whose end x is contained in Σ . Then*

$$\Sigma \cap \overline{B} = \bigcup_{y \in B \cap (V \cup D)} [x, y].$$

Proof. We may assume without loss of generality that $x \in V$. Let $V_0 = V \setminus (V \cap B)$ and $D_0 = D \setminus (D \cap B)$. Then V_0 is a semistable vertex set of (X, D_0) : indeed, every connected component of $X^{\text{an}} \setminus (V \cup D)$ is either a connected component of $X^{\text{an}} \setminus (V_0 \cup D_0)$ or is contained in B , so $X^{\text{an}} \setminus (V_0 \cup D_0)$ is a disjoint union of open balls and finitely many generalized open annuli. Let $\Sigma_0 = \Sigma(X, V_0 \cup D_0)$, so $x \in \Sigma_0$ but $B \cap \Sigma_0 = \emptyset$ by Proposition 3.12(1). By Lemma 4.6 we have that B is a connected component of $X^{\text{an}} \setminus \Sigma_0$, so the Lemma now follows from Lemma 3.15 as applied to $\Sigma = \Sigma_0$, $W = V \cap B$, and $E = D \cap B$. ■

Proof of Theorem 4.13. Note that $\varphi(\Sigma') \subset \Sigma$ implies $\varphi(D') \subset D$. Let V (resp. V') be a vertex set for Σ (resp. Σ'). We claim that

$$\varphi^{-1}(\Sigma) = \Sigma' \cup \bigcup_{x' \in \varphi^{-1}(V \cup D)} [x', \tau_{\Sigma'}(x')],$$

which by Lemma 3.15 is the minimal skeleton of $(X', \varphi^{-1}(D))$ containing Σ' and $\varphi^{-1}(V)$.

Let B' be a connected component of $X'^{\text{an}} \setminus \Sigma'$, let $x' \in \Sigma'$ be its end, and let $x = \varphi(x') \in \Sigma$. Then $B = \varphi(B')$ is an open ball and x is its end: if $X \not\cong \mathbf{P}^1$, then this follows directly from Lemma 4.14, and if $X \cong \mathbf{P}^1$, then $B \subset X^{\text{an}} \setminus \{z\}$, so $\varphi(B) \neq X^{\text{an}}$ and therefore B is an open ball in this case as well. Hence

$$\Sigma \cap B = \bigcup_{y \in B \cap (V \cup D)} [x, y]$$

by Lemma 4.16, so

$$\varphi^{-1}(\Sigma) \cap B' = \bigcup_{y' \in B' \cap \varphi^{-1}(V \cup D)} [x', y']$$

by Proposition 4.8. ■

Remark 4.17. When $X \cong \mathbf{P}^1$, the extra hypothesis on Σ' in Theorem 4.13 is necessary. Indeed, let $\varphi : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ be a finite morphism and let $x' \in \mathbf{P}^{1,\text{an}}$ be a type-2 point such that $\varphi^{-1}(\varphi(x'))$ has more than one element. Then $\Sigma = \{\varphi(x')\}$ is a skeleton containing the image of the skeleton $\Sigma' = \{x'\}$, but $\varphi^{-1}(\Sigma)$ is not a skeleton.

Corollary 4.18. *Let Σ (resp. Σ') be a skeleton of (X, D) (resp. (X', D')). There exists a skeleton Σ_1 of $(X, D \cup \varphi(D'))$ such that $\Sigma_1 \supset \Sigma \cup \varphi(\Sigma')$ and such that $\varphi^{-1}(\Sigma_1)$ is a skeleton of $(X', \varphi^{-1}(D \cup \varphi(D')))$. Moreover, there is a minimal such Σ_1 with respect to inclusion.*

Proof. If $X \not\cong \mathbf{P}^1$ then this is an immediate consequence of Proposition 4.11 and Theorem 4.13: if Σ_1 is the minimal skeleton of $(X, D \cup \varphi(D'))$ containing $\Sigma \cup \varphi(\Sigma')$, then $\varphi^{-1}(\Sigma_1)$ is a skeleton of $(X', \varphi^{-1}(D \cup \varphi(D')))$. Suppose then that $X \cong \mathbf{P}^1$. Let $x' \in \Sigma'$ be a type-2 point, let $x = \varphi(x')$, and let $x', x'_1, x'_2, \dots, x'_n$ be the points of X' mapping to x . Let

$$\Sigma'_1 = \Sigma' \cup \bigcup_{i=1}^n [x'_i, \tau_{\Sigma'}(x'_i)].$$

This is the minimal skeleton of (X', D') containing Σ' and $\varphi^{-1}(x)$ by Lemma 3.15(3). Let Σ_1 be the minimal skeleton of $(X, D \cup \varphi(D'))$ containing $\Sigma \cup \varphi(\Sigma'_1)$. Then $\varphi^{-1}(\Sigma_1)$ is a skeleton of $(X', \varphi^{-1}(D \cup \varphi(D')))$ by Theorem 4.13. If Σ_2 is a skeleton of $(X, D \cup \varphi(D'))$ such that $\Sigma_2 \supset \Sigma \cup \varphi(\Sigma')$ and $\varphi^{-1}(\Sigma_2)$ is a skeleton of $(X', \varphi^{-1}(D \cup \varphi(D')))$, then $\varphi^{-1}(\Sigma_2)$ contains each geodesic $[x'_i, \tau_{\Sigma'}(x'_i)]$, so $\varphi^{-1}(\Sigma_2)$ contains Σ'_1 and therefore $\Sigma_2 \supset \Sigma_1$. ■

Remark 4.19. Suppose that $\varphi : X' \rightarrow X$ is a finite morphism such that $\varphi^{-1}(D) = D'$ and $\varphi^{-1}(V) = V'$. Let $\Sigma = \Sigma(X, V \cup D)$ and $\Sigma' = \Sigma(X', V' \cup D')$. In this case we can describe the skeleton Σ_1 of Corollary 4.18 more explicitly, as follows. Let e' be an open edge of Σ' with respect to the given choice of vertex set and let $A' = \tau^{-1}(e')$. This is a connected component of $X'^{\text{an}} \setminus (V' \cup D')$. Let A be the connected component of $X^{\text{an}} \setminus (V \cup D)$ containing $\varphi(A')$. Since A' is a connected component of $\varphi^{-1}(A)$, the map $A' \rightarrow A$ is finite, hence surjective. If A' is a punctured open ball, then so is A , so by Lemma 4.2, φ maps e' homeomorphically with nonzero integer expansion factor onto $e = \Sigma(A)$. If A' is an open annulus, then A could be an open annulus or an open ball; in the former case we again have e' mapping homeomorphically onto $e = \Sigma(A)$ with nonzero integer expansion factor. If A is an open ball, then by Proposition 4.9, e' maps onto a geodesic segment in A ; clearly both vertices of e' map to the end of A , so e' “goes straight and doubles back” under φ .

It follows from this and Proposition 4.12(2) that $\varphi(\Sigma')$ is equal to Σ union a finite number of geodesic segments T_1, \dots, T_r contained in open balls $B_1, \dots, B_r \subset X^{\text{an}} \setminus (V \cup D)$ and with one endpoint at elements of V . These are precisely the images of the edges of Σ' not mapping to edges of Σ . By Lemma 3.15, $\Sigma \cup (T_1 \cup \dots \cup T_r)$ is again a skeleton of X , so in this case

$$\Sigma_1 = \varphi(\Sigma') = \Sigma \cup (T_1 \cup \dots \cup T_r)$$

(the inverse image of Σ_1 is a skeleton by Theorem 4.13).

4.20. Tangent directions and morphisms of curves. Our next goal is to prove that a finite morphism of curves induces a finite harmonic morphism of metrized complexes of curves for a suitable choice of skeleta. First we need to formulate and prove a relative version of Theorem 3.21. Let X, X' be smooth, proper, connected K -curves.

Definition 4.21. A continuous function $\varphi : X'^{\text{an}} \rightarrow X^{\text{an}}$ is *piecewise affine* provided that, for every geodesic segment $\alpha : [a, b] \hookrightarrow X'^{\text{an}}$, the composition $\varphi \circ \alpha$ is piecewise affine with integer expansion factors in the sense of Definition 4.4.

Let $\varphi : X'^{\text{an}} \rightarrow X^{\text{an}}$ be a piecewise affine function, let $x' \in X'^{\text{an}}$, and let $x = \varphi(x') \in X^{\text{an}}$. Let $v' \in T_{x'}$ and let $\alpha : [a, b] \hookrightarrow X'^{\text{an}}$ be a geodesic segment representing v' (so $a = -\infty$ if x' has type 1). Taking b small enough, we can assume that $\varphi \circ \alpha$ is an embedding with integer expansion factor m . Let $v \in T_x$ be the tangent direction represented by $\varphi \circ \alpha$. We define

$$d\varphi(x') : T_{x'} \rightarrow T_x \quad \text{by} \quad d\varphi(x')(v') = v;$$

this is independent of the choice of α . We call the expansion factor m the *outgoing slope of φ in the direction v'* and we write $m = d_{v'}\varphi(x')$.

Definition 4.22. A piecewise affine function $\varphi : X'^{\text{an}} \rightarrow X^{\text{an}}$ is *harmonic* at a point $x' \in X'^{\text{an}}$ provided that, for all $v \in T_{\varphi(x')}$, the integer

$$\sum_{\substack{v' \in T_{x'} \\ d\varphi(x')(v') = v}} d_{v'}\varphi(x')$$

is independent of the choice of v .

Let $\varphi : X' \rightarrow X$ be a finite morphism of smooth, proper, connected K -curves, let $x' \in X'^{\text{an}}$ be a type-2 point, and let $x = \varphi(x')$. Let C_x and $C_{x'}$ denote the smooth proper connected k -curves with function fields $\mathcal{H}(x)$ and $\mathcal{H}(x')$, respectively. We denote by $\varphi_{x'}$ the induced morphism $C_{x'} \rightarrow C_x$. We have the following relative version of the slope formula of Theorem 3.21:

Theorem 4.23. Let $\varphi : X' \rightarrow X$ be a finite morphism of smooth, proper, connected K -curves.

- (1) The analytification $\varphi : X'^{\text{an}} \rightarrow X^{\text{an}}$ is piecewise affine and harmonic.
- (2) Let $x' \in X'^{\text{an}}$ be a type-2 point, let $x = \varphi(x')$, let $v' \in T_{x'}$, let $v = d\varphi(v') \in T_x$, and let $\xi_{v'} \in C_{x'}$ and $\xi_v \in C_x$ be the closed points associated (3.22.1) to v' and v , respectively. Then $\varphi_{x'}(\xi_{v'}) = \xi_v$, and the ramification degree of $\varphi_{x'}$ at $\xi_{v'}$ is equal to $d_{v'}\varphi(x')$.
- (3) Let $x' \in X'^{\text{an}}$ be a type-1 point and let $v' \in T_{x'}$ be the unique tangent direction. Then $d_{v'}\varphi(x')$ is the ramification degree of φ at x' .

Proof. First we claim that φ is piecewise affine. Let $[x', y'] \subset X'^{\text{an}}$ be a geodesic segment. Suppose first that x' and y' have type 2 or 3. Then there exists a skeleton Σ' of X' containing $[x', y']$ by [BPR11, Corollaries 5.56 and 5.64]. From this we easily reduce to the case that $[x', y']$ is an edge of Σ' with respect to some vertex set; now the claim follows from Lemma 4.2 and Proposition 4.9 as in the proof of Proposition 4.11. If x' has type 1 or 4, then there is an open neighborhood B of $x = \varphi(x')$ and an open neighborhood $B' \subset \varphi^{-1}(B)$ of x' such that B and B' are open balls. Shrinking B' if necessary we can assume that $y' \notin B'$, so the end z' of B' is contained in $[x', y']$ (removing z' disconnects X'^{an}). The restriction of φ to $[x', z']$ is piecewise affine by Lemma 4.5, so this proves the claim.

Next we prove (2). By functoriality of the reduction map, for any non-zero rational function f on X , we have $\varphi_{x'}^*(\tilde{f}_x) = \varphi^*(f)_{x'}$. By the slope formula (Theorem 3.21), the point ξ_v corresponds to the discrete valuation on $\tilde{\mathcal{H}}(x)$ given by $\text{ord}_{\xi_v}(\tilde{f}_x) = d_v F(x)$, where $F = -\log |f|$. Similarly, the slope formula applied to $\varphi^*(f)$ on X' gives $\text{ord}_{\xi_{v'}}(\varphi^*(f)_{x'}) = d_{v'}\varphi(x')d_v F(x)$. Therefore, we get

$$\text{ord}_{\xi_{v'}}(\varphi_{x'}^*(\tilde{f}_x)) = d_{v'}\varphi(x')\text{ord}_{\xi_v}(\tilde{f}_x).$$

Since any non-zero element of $\tilde{\mathcal{H}}(x)$ is of the form \tilde{f}_x for some non-zero rational function f on X , this shows that the center of the valuation $\xi_{v'}$ on $\tilde{\mathcal{H}}(x) \xrightarrow{\varphi_{x'}^*} \tilde{\mathcal{H}}(x')$ coincides with the center of ξ_v , that is $\varphi_{x'}(\xi_{v'}) = \xi_v$, and at the same time, the ramification degree of $\varphi_{x'}$ at $\xi_{v'}$ is equal to $d_{v'}\varphi(x')$.

The proof of (3) proceeds in the same way as (2), applying the slope formula to a non-zero rational function f on X with a zero at $x = \varphi(x')$, and to $\varphi^*(f)$ on X' . \blacksquare

4.24. Morphisms of curves induce morphisms of metrized complexes. More precisely, this occurs when the morphism of curves respects a choice of triangulations, in the following sense.

Definition 4.25. Let $(X, V \cup D)$ and $(X', V' \cup D')$ be triangulated punctured curves. A finite morphism $\varphi : (X', V' \cup D') \rightarrow (X, V \cup D)$ is a finite morphism $\varphi : X' \rightarrow X$ such that $\varphi^{-1}(D) = D'$, $\varphi^{-1}(V) = V'$, and $\varphi^{-1}(\Sigma(X, V \cup D)) = \Sigma(X', V' \cup D')$ (as sets).

Corollary 4.26. Let $\varphi : X' \rightarrow X$ be a finite morphism of smooth, connected, proper K -curves, let $D \subset X(K)$ be a finite set, and let $D' = \varphi^{-1}(D)$. There exists a strongly semistable vertex set V of (X, D) such that $V' = \varphi^{-1}(V)$ is a strongly semistable vertex set of (X', D') and such that

$$\Sigma(X', V' \cup D') = \varphi^{-1}(\Sigma(X, V \cup D)).$$

In particular, φ extends to a finite morphism of triangulated punctured curves.

Proof. By Corollary 4.18, there is a skeleton Σ of (X, D) such that $\Sigma' = \varphi^{-1}(\Sigma)$ is a skeleton of (X', D') . Let V'_0 be a strongly semistable vertex set of Σ' , let V be a strongly semistable vertex set of Σ containing $\varphi(V'_0)$, and let $V' = \varphi^{-1}(V)$. Then V' is again a strongly semistable vertex set of Σ' . \blacksquare

4.27. Let $\varphi : (X', V' \cup D') \rightarrow (X, V \cup D)$ be a finite morphism of triangulated punctured curves. Let $\Sigma = \Sigma(X, V \cup D)$ and $\Sigma' = \Sigma(X', V' \cup D')$. Let e' be an open edge of Σ' and let $e = \varphi(e')$, an open edge of Σ . The image of the annulus $A' = \tau^{-1}(e')$ is contained in $A = \tau^{-1}(e)$, so the restriction of φ to e' is an embedding with integer expansion factor by Lemma 4.2(2). It is clear from this and Theorem 4.23 that $\varphi|_{\Sigma'} : \Sigma' \rightarrow \Sigma$ is a harmonic (V', V) -morphism of Λ -metric graphs. For $x' \in V'$ with image $x = \varphi(x')$ we have a finite morphism of k -curves $\varphi_{x'} : C_{x'} \rightarrow C_x$. It now follows from Theorem 4.23 that these extra data enrich $\varphi|_{\Sigma'} : \Sigma' \rightarrow \Sigma$ with the structure of a finite harmonic morphism of Λ -metrized complexes of curves:

Corollary 4.28. *Let $\varphi : (X', V' \cup D') \rightarrow (X, V \cup D)$ be a finite morphism of triangulated punctured curves. Then φ naturally induces a finite harmonic morphism of Λ -metrized complexes of curves*

$$\Sigma(X', V' \cup D') \longrightarrow \Sigma(X, V \cup D).$$

Remark 4.29. More generally, suppose that $(X, V \cup D)$ and $(X', V' \cup D')$ are triangulated punctured K -curves and that $\varphi : X' \rightarrow X$ is a finite morphism such that $\varphi^{-1}(D) = D'$ and $\varphi^{-1}(V) \subset V'$. Then $\varphi^{-1}(\Sigma) \subset \Sigma'$ by Proposition 4.12(2). One can show using Theorem 4.23 that the map $\Sigma' \rightarrow X^{\text{an}}$ followed by the retraction $X^{\text{an}} \rightarrow \Sigma$ is a (not necessarily finite) harmonic morphism of metric graphs $\Sigma' \rightarrow \Sigma$.

4.30. Tame coverings of triangulated curves. Many of the results in this paper involve constructing a curve X' and a morphism $\varphi : X' \rightarrow X$ inducing a given morphism of skeleta as in Corollary 4.28. For these purposes it is useful to introduce some mild restrictions on the morphism φ . Fix a triangulated punctured curve $(X, V \cup D)$ with skeleton $\Sigma = \Sigma(X, V \cup D)$, regarded as a metrized complex of curves as in (3.22).

Definition 4.31. Let $(X', V' \cup D')$ be a triangulated punctured curve with skeleton $\Sigma' = \Sigma(X', V' \cup D')$ and let $\varphi : (X', V' \cup D') \rightarrow (X, V \cup D)$ be a finite morphism. We say that φ is a *tame covering* of $(X, V \cup D)$ provided that:

- (1) D contains the branch locus of φ ,
- (2) if $\text{char}(k) = p > 0$, then for every edge $e' \in E(\Sigma')$ the expansion factor $d_{e'}(\varphi)$ is not divisible by p , and
- (3) $\varphi_{x'}$ is separable (= generically étale) for all $x' \in V'$.

Remark 4.32.

- (1) Since D is finite, it follows from (1) and Theorem 4.23(3) that a tame covering of $(X, V \cup D)$ is a tamely ramified morphism of curves.
- (2) If Σ has at least one edge, then (2) implies (3).
- (3) Let $\varphi : (X', V' \cup D') \rightarrow (X, V \cup D)$ be a tame covering, let $S \subset \Sigma$ be a finite set of type-2 points, and let $S' = \varphi^{-1}(S)$. Then $S' \subset \Sigma(X', V' \cup D')$ is also a finite set of type-2 points, so φ also defines a tame covering $(X', S' \cup V' \cup D') \rightarrow (X, S \cup V \cup D)$.

Lemma 4.33. *Let $\varphi : (X', V' \cup D') \rightarrow (X, V \cup D)$ be a tame covering and let $\Sigma = \Sigma(X, V \cup D)$ and $\Sigma' = \Sigma(X', V' \cup D')$. Then $\varphi|_{\Sigma'} : \Sigma' \rightarrow \Sigma$ is a tame covering of metrized complexes of curves.*

Proof. Let Γ (resp. Γ') be the metric graph underlying Σ (resp. Σ'). We must show that $\varphi|_{\Gamma'}$ is generically étale. Let $R = \sum_{x' \in V' \cup D'} R_{x'}(x')$ be the ramification divisor of $\varphi|_{\Gamma'}$ as defined in (2.14.1), and let $S = \sum_{x' \in D'} S_{x'}(x')$ be the ramification divisor of $\varphi : X' \rightarrow X$. We will show that $R = S$. Since $\varphi_{x'}$ is generically étale for all $x' \in V'$, we see from (2.14.1) and the Riemann-Hurwitz formula as applied to $\varphi_{x'}$ that $R_{x'} \geq 0$. If $x' \in D'$, then there is a unique edge e' adjacent to x' and $d_{e'}(\varphi) = d_{x'}(\varphi)$, so

$$R_{x'} = 2 d_{x'}(\varphi) - 2 - d_{x'}(\varphi) + 1 = d_{x'}(\varphi) - 1 = S_{x'},$$

where the final equality is Theorem 4.23(3). Therefore it is enough to show that $\deg(R) = \deg(S)$. By (2.14.2) we have $K_{\Gamma'} = (\varphi|_{\Gamma'})^*(K_{\Gamma}) + R$, so counting degrees gives

$$\deg(R) = \deg(\varphi|_{\Gamma'}) (2 - 2g(\Gamma)) - (2 - 2g(\Gamma')).$$

The Lemma now follows from the equalities $\deg(\varphi) = \deg(\varphi|_{\Gamma'})$, $g(\Gamma) = g(X)$, $g(\Gamma') = g(X')$, and the Riemann-Hurwitz formula as applied to $\varphi : X' \rightarrow X$. ■

Remark 4.34. We showed in the proof of Lemma 4.33 that the ramification divisor of $\varphi|_{\Gamma'}$ coincides with the ramification divisor of φ . Moreover, it follows from Proposition 2.25 that for every $x' \in V'$, every ramification point $\bar{x}' \in C_{x'}$ is the reduction of a tangent direction in Γ' . In other words, for a

tame covering all ramification points of all residue curves are “visible” in the morphism of underlying metric graphs $\Gamma' \rightarrow \Gamma$.

Proposition 4.35. *Let $\varphi : (X', V' \cup D') \rightarrow (X, V \cup D)$ be a finite morphism of triangulated punctured curves. Then φ is a tame covering if and only if*

- (1) *D contains the branch locus of φ and*
- (2) *$\varphi_{x'} : C_{x'} \rightarrow C_{\varphi(x)}$ is tamely ramified for every type-2 point $x' \in X'^{\text{an}}$.*

Moreover, if φ is a tame covering and $x' \in X'^{\text{an}}$ is a type-2 point not contained in Σ' , then φ is an isomorphism in a neighborhood of x' .

Proof. It is clear that conditions (1)–(2) imply that φ is a tame covering, so suppose that φ is a tame covering and $x' \in X'^{\text{an}}$ is a type-2 point. We have the following cases:

- If $x' \in V'$, then $\varphi_{x'}$ is tamely ramified by Lemma 4.33 and Proposition 2.25.
- Suppose that x' is contained in the interior of an edge e' of Σ' . Since φ is a tame covering of $(X, (V \cup \{\varphi(y')\}) \cup D)$ by Remark 4.32(3), we are reduced to the case treated above.
- Suppose that $x' \notin \Sigma'$ but $y' = \tau(x')$ is contained in V' . Let \mathfrak{X} (resp. \mathfrak{X}') be the semistable formal model of X (resp. X') corresponding to the semistable vertex set V (resp. V') and let φ denote the unique finite morphism $\mathfrak{X}' \rightarrow \mathfrak{X}$ extending $\varphi : X' \rightarrow X$ (see Theorem 5.13). Let $\bar{x}' = \text{red}(x') \in \mathfrak{X}'(k)$ and let $x = \varphi(x')$ and $\bar{x} = \text{red}(x) = \varphi_k(\bar{x}')$. The connected component B' of x' in $X'^{\text{an}} \setminus \Sigma'$ is equal to $\text{red}^{-1}(\bar{x}')$, and the connected component B of x in $X^{\text{an}} \setminus \Sigma$ is equal to $\text{red}^{-1}(\bar{x})$. Since $\Sigma \cap B = \emptyset$, φ_k is not branched over \bar{x} by Lemma 4.33 and Proposition 2.25. Therefore $\varphi^{-1}(B)$ is a disjoint union of $\deg(\varphi)$ open balls (one of which is B') mapping isomorphically onto B . In particular, φ is an isomorphism in a neighborhood of x' .
- Suppose that $x' \notin \Sigma'$ but $y' = \tau(x')$ is contained in the interior of an edge of Σ' . We claim that φ is an isomorphism in a neighborhood of x' . Let $y = \varphi(y') \in \Sigma$. Then φ is a tame covering of $(X, (V \cup \{y\}) \cup D)$, so replacing V with $V \cup \{y\}$ and V' with $V' \cup \varphi^{-1}(y)$ (see Remark 4.32(3)), we are reduced to the previous case. ■

Remark 4.36. It follows from Proposition 4.35 that if φ is a tame covering then the set-theoretic branch locus of $\varphi : X'^{\text{an}} \rightarrow X^{\text{an}}$ (i.e. the set of all points in X^{an} with fewer than $\deg(\varphi)$ points in its fiber) is contained in Σ . See [Fab11a, Fab11b] for more on the topic of the Berkovich ramification locus in the case of self-maps of \mathbf{P}^1 .

5. APPLICATIONS TO MORPHISMS OF SEMISTABLE MODELS

In this section we show how Section 4 formally implies a large part of the results of Liu [Liu06] on simultaneous semistable reduction of morphisms of curves (see also [Col03]) over an arbitrary valued field. This section is for the most part logically independent of the rest of this paper (although we do use certain facts about the relationship with formal models in an important way in the sequel), but we feel that the “skeletal” point of view on simultaneous semistable reduction is enlightening (see Remark 5.23). To this end, we let K_0 be any field equipped with a nontrivial non-Archimedean valuation $\text{val} : K_0 \rightarrow \mathbf{R} \cup \{\infty\}$. Its valuation ring will be denoted R_0 , its maximal ideal \mathfrak{m}_{R_0} , and its residue field k_0 .

Let X be a smooth, proper, geometrically connected algebraic K_0 -curve. By a (strongly) semistable R_0 -model of X we mean a flat, integral, proper relative curve $\mathcal{X} \rightarrow \text{Spec}(R_0)$ whose special fiber \mathcal{X}_{k_0} is a (strongly) semistable curve (i.e. \mathcal{X}_{k_0} is a reduced curve with at worst ordinary double point singularities; it is strongly semistable if its irreducible components are smooth) and whose generic fiber is equipped with an isomorphism $\mathcal{X}_{K_0} \cong X$. By properness of \mathcal{X} , any K_0 -point $x \in X(K_0)$ extends in a unique way to an R_0 -point $x \in \mathcal{X}(R_0)$; the special fiber of this point is the *reduction* of x and is denoted $\text{red}(x) \in \mathcal{X}(k_0)$. Let $D \subset X(K_0)$ be a finite set. A semistable model \mathcal{X} of X is a *semistable R_0 -model of (X, D)* provided that the points of D reduce to distinct smooth points of \mathcal{X}_{k_0} .

The model \mathcal{X} is a *stable R_0 -model* of (X, D) provided that every rational (resp. genus-1) component of the normalization \mathcal{X}_k contains at least three points (resp. one point) mapping to a singular point of \mathcal{X}_k or to the reduction of a point of D .

If $K_0 = K$ is complete and algebraically closed, we define a (*strongly*) *semistable formal R -model* of X to be an integral, proper, admissible formal R -curve \mathfrak{X} whose analytic generic fiber \mathfrak{X}_K is equipped with an isomorphism to X^{an} and whose special fiber \mathfrak{X}_k is a (strongly) semistable curve. There is a natural map of sets $\text{red} : X^{\text{an}} \rightarrow \mathfrak{X}_k$, called the *reduction*; it is surjective and anti-continuous in the sense that the inverse image of a Zariski-open subset of \mathfrak{X}_k is a closed subset of X^{an} (and vice-versa). Using the reduction map, for a finite set $D \subset X(K)$ we define *semistable* and *stable formal R -models* of (X, D) as above.

As we will be passing between formal and algebraic R -models of K -curves, it is worth stating the following lemma; see [BPR11, Remark 5.30(2)].

Lemma 5.1. *Let X be a smooth, proper, connected K -curve. The ϖ -adic completion functor defines an equivalence from the category of semistable R -models of X to semistable formal R -models of X .*

The inverse of the ϖ -adic completion functor of Lemma 5.1 will be called *algebraization*.

5.2. Descent to a general ground field. We will use the following lemmas to descend the geometric theory of Section 4 to the valued field K_0 . Fix an algebraic closure \bar{K}_0 of K_0 and a valuation val on \bar{K}_0 extending the given valuation on K_0 . For any field $K_1 \subset \bar{K}_0$ we consider K_1 as a valued field with respect to the restriction of val , and we write R_1 for the valuation ring of K_1 . Let K be the completion of \bar{K}_0 with respect to val . This field is algebraically closed by [BGR84, Proposition 3.4.1/3].

Lemma 5.3. *Let X_0 be a smooth, proper, geometrically connected curve over K_0 and let $X = X_0 \otimes_{K_0} K$. Let K_0^{sep} be the separable closure of K_0 in K . Then the image of $X_0(K_0^{\text{sep}})$ under the natural inclusion*

$$X_0(K_0^{\text{sep}}) \subset X_0(K) = X(K) \subset X^{\text{an}}$$

is dense in X^{an} .

Proof. Let $\hat{K}_0 \subset K$ denote the completion of K_0 and let $(\hat{K}_0)^{\text{sep}} \subset K$ be its separable closure. By [BGR84, Proposition 3.4.1/6], $(\hat{K}_0)^{\text{sep}}$ is dense in K . Let $\hat{K}_1 \subset (\hat{K}_0)^{\text{sep}}$ be a finite, separable extension of \hat{K}_0 . By Krasner's lemma (see §3.4.2 of loc. cit.), there exists a finite, separable extension K_1/K_0 contained in K_0^{sep} which is dense in \hat{K}_1 . It follows that K_0^{sep} is dense in $(\hat{K}_0)^{\text{sep}}$, so K_0^{sep} is dense in K . Since $\mathbf{P}_K^1(K)$ is dense in $\mathbf{P}_K^{1,\text{an}}$ and the subspace topology on $\mathbf{P}_K^1(K) \subset \mathbf{P}_K^{1,\text{an}}$ coincides with the ultrametric topology, this proves the lemma for $X_0 = \mathbf{P}_{K_0}^1$.

For general X_0 , choose a finite, generically étale morphism $\varphi : X_0 \rightarrow \mathbf{P}_{K_0}^1$. Let $D \subset \mathbf{P}_K^1(K)$ be the branch locus, so $\varphi : X^{\text{an}} \setminus \varphi^{-1}(D) \rightarrow \mathbf{P}_K^{1,\text{an}} \setminus D$ is étale, hence open by [Ber93, Corollary 3.7.4]. It follows that if $U \subset X^{\text{an}}$ is a nonempty open set, then $\varphi(U) \setminus D$ contains a nonempty open subset of $\mathbf{P}_K^{1,\text{an}} \setminus D$. Let $x \in \mathbf{P}_K^1(K_0^{\text{sep}})$ be a point contained in $\varphi(U) \setminus D$. Then $\varphi^{-1}(x) \subset X_0(K_0^{\text{sep}})$ and $\varphi^{-1}(x) \cap U \neq \emptyset$. ■

Lemma 5.4. *Let X_0 be a smooth, proper, geometrically connected curve over K_0 , let $D \subset X_0(K_0)$ be a finite subset, let $X = X_0 \otimes_{K_0} K$, and let \mathcal{X} be a semistable model of (X, D) . There exists a finite, separable extension K_1 of K_0 and a semistable model \mathcal{X}_1 of $X_1 = X_0 \otimes_{K_0} K_1$ with respect to D such that $\mathcal{X}_1 \otimes_{R_1} R \cong \mathcal{X}$.*

Proof. Let \mathfrak{X} be the ϖ -adic completion of \mathcal{X} . For $\bar{x} \in \mathfrak{X}_k(k)$ the subset $\text{red}^{-1}(\bar{x}) \subset X^{\text{an}}$ is open in X^{an} . By Lemma 5.3, there exists a point $x \in X_0(K_0^{\text{sep}})$ with $\text{red}(x) = \bar{x}$, so after passing to a finite, separable extension of K_0 if necessary, we may enlarge $D \subset X_0(K_0)$ to assume that \mathcal{X} is a *stable* model of (X, D) . On the other hand, by the stable reduction theorem (i.e. the properness of the Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}$ parameterizing stable marked curves) there exists a stable model \mathcal{X}_0 of (X_0, D_0) after potentially passing to a finite, separable extension of K_0 . By uniqueness of stable models we have $\mathcal{X}_0 \otimes_{R_0} R \cong \mathcal{X}$. ■

Lemma 5.5. *Let X_0, X'_0 be smooth, proper, geometrically connected K_0 -curves and let $\varphi : X'_0 \rightarrow X_0$ be a finite morphism. Let \mathcal{X}_0 (resp. \mathcal{X}'_0) be a semistable model of X_0 (resp. X'_0), let $X = X_0 \otimes_{K_0} K$ and $\mathcal{X} = \mathcal{X}_0 \otimes_{R_0} R$ (resp. $X' = X'_0 \otimes_{K_0} K$ and $\mathcal{X}' = \mathcal{X}'_0 \otimes_{R_0} R$), and suppose that $\varphi_K : X' \rightarrow X$ extends to a morphism $\mathcal{X}' \rightarrow \mathcal{X}$. Then φ extends to a morphism $\mathcal{X}'_0 \rightarrow \mathcal{X}_0$ in a unique way.*

Proof. The morphism φ gives rise to a section $X'_0 \rightarrow X_0 \times_{K_0} X'_0$ of $\text{pr}_2 : X_0 \times_{K_0} X'_0 \rightarrow X'_0$. Let \mathcal{Z} be the schematic closure of the image of X'_0 in $\mathcal{X}_0 \times_{R_0} \mathcal{X}'_0$. It is enough to show that $\text{pr}_2 : \mathcal{Z} \rightarrow \mathcal{X}'_0$ is an isomorphism. Since schematic closure respects flat base change, the schematic closure of the image of X' in $\mathcal{X} \times_R \mathcal{X}'$ is equal to $\mathcal{Z} \otimes_{R_0} R$. By properness of \mathcal{X}' , the image of \mathcal{X}' in $\mathcal{X} \times_R \mathcal{X}'$ is then equal to $\mathcal{Z} \otimes_{R_0} R$, so $\text{pr}_2 : \mathcal{Z} \otimes_{R_0} R \rightarrow \mathcal{X}'$ is an isomorphism. Since $R_0 \rightarrow R$ is faithfully flat, this implies that $\text{pr}_2 : \mathcal{Z} \rightarrow \mathcal{X}'_0$ is an isomorphism. \blacksquare

5.6. The relation between semistable models and skeleta. Let X be a smooth, proper, connected K -curve. The following theorem is due to Berkovich and Bosch-Lütkebohmert; see [BPR11, Theorem 5.34] for more precise references.

Theorem 5.7. *Let \mathfrak{X} be a semistable formal model of X and let $\bar{x} \in \mathfrak{X}_k$ be a point. Then*

- (1) \bar{x} is a generic point if and only if $\text{red}^{-1}(\bar{x})$ consists of a single type-2 point of X^{an} .
- (2) \bar{x} is a smooth closed point if and only if $\text{red}^{-1}(\bar{x})$ is an open ball.
- (3) \bar{x} is an ordinary double point if and only if $\text{red}^{-1}(\bar{x})$ is an open annulus.

It follows from Theorem 5.7 that if \mathfrak{X} is a semistable formal model of X , then

$$V(\mathfrak{X}) = \{ \text{red}^{-1}(\bar{\zeta}) : \zeta \in \mathfrak{X}_k \text{ is a generic point} \}$$

is a semistable vertex set of X : indeed,

$$(5.7.1) \quad X \setminus V(\mathfrak{X}) = \coprod_{\substack{\bar{x} \in \mathfrak{X}_k \\ \text{singular}}} \text{red}^{-1}(\bar{x}) \cup \coprod_{\substack{\bar{x} \in \mathfrak{X}_k \\ \text{smooth}}} \text{red}^{-1}(\bar{x})$$

is a disjoint union of open balls and finitely many open annuli. The following folklore theorem is proved in [BPR11, Theorem 5.38].

Theorem 5.8. *Let X be a smooth, proper, connected algebraic K -curve and let $D \subset X(K)$ be a finite set of closed points. The association $\mathfrak{X} \mapsto V(\mathfrak{X})$ sets up a bijection between the set of (strongly) semistable formal models of (X, D) (up to isomorphism) and the set of (strongly) semistable vertex sets of (X, D) .*

5.9. Let \mathfrak{X} be a semistable formal model of (X, D) . One can construct the metrized complex of curves $\Sigma(X, V(\mathfrak{X}) \cup D)$ directly from \mathfrak{X} , as follows. Let V be the set of irreducible components of \mathfrak{X}_k , regarded as a set of vertices of a graph. For $x \in V$ we will write C_x to denote the corresponding component of \mathfrak{X}_k . To every double point $\bar{x} \in \mathfrak{X}_k$ we associate an edge e connecting the irreducible components of \mathfrak{X}_k containing \bar{x} (this is a loop edge if there is only one such component); if C_x is such a component, then we set $\text{red}_x(e) = \bar{x}$. The completed local ring of \mathfrak{X} at \bar{x} is isomorphic to $R[[s, t]]/(st - \pi)$ for some $\pi \in \mathfrak{m}_R \setminus \{0\}$; we define the length of e to be $\ell(e) = \text{val}(\pi)$. We connect a point $x \in D$ to the vertex corresponding to the irreducible component of \mathfrak{X}_k containing $\text{red}(x)$. These data define a metrized complex of curves $\Sigma(\mathfrak{X}, D)$. By (5.7.1), the graphs underlying $\Sigma(\mathfrak{X}, D)$ and $\Sigma(X, V(\mathfrak{X}) \cup D)$ are naturally isomorphic. The edge lengths coincide — see for instance [BPR11, Proposition 5.37] — and the residue curves of the two metrized complexes are naturally isomorphic by [Ber90, Proposition 2.4.4]. From this it is straightforward to verify that $\Sigma(\mathfrak{X}, D)$ and $\Sigma(X, V(\mathfrak{X}) \cup D)$ are identified as metrized complexes of curves.

In particular, a semistable formal model \mathfrak{X} of (X, D) is stable if and only if $V(\mathfrak{X})$ is a stable vertex set of (X, D) .

As another consequence of Theorem 5.8, we obtain the following compatibility of skeleta and extension of scalars.

Proposition 5.10. *Let X be a smooth, proper, connected K -curve, let K' be a complete and algebraically closed valued field extension of K , and let $\pi : X_{K'}^{\text{an}} \rightarrow X_K^{\text{an}}$ be the canonical map. If \mathfrak{X} is a semistable formal model of X , then*

- (1) π maps $V(\mathfrak{X}_{R'})$ bijectively onto $V(\mathfrak{X})$, and
- (2) π maps $\Sigma(X_{K'}, V(\mathfrak{X}_{R'}) \cup D)$ bijectively onto $\Sigma(X, V(\mathfrak{X}) \cup D)$, with

$$\pi : \Sigma(X_{K'}, V(\mathfrak{X}_{R'}) \cup D) \xrightarrow{\sim} \Sigma(X, V(\mathfrak{X}) \cup D)$$

an isomorphism of augmented metric graphs.

Proof. The reduction map is compatible with extension of scalars, in that the following square commutes

$$\begin{array}{ccc} X_{K'}^{\text{an}} & \xrightarrow{\text{red}} & \mathfrak{X}_{k'} \\ \pi \downarrow & & \downarrow \bar{\pi} \\ X^{\text{an}} & \xrightarrow{\text{red}} & \mathfrak{X}_k \end{array}$$

with k' the residue field of K' . The first assertion is immediate because the extension of scalars morphism $\bar{\pi} : \mathfrak{X}_{k'} \rightarrow \mathfrak{X}_k$ is a bijection on generic points. It is also a bijection on singular points, and for any node $\bar{x}' \in \mathfrak{X}_{k'}$ the open annulus $A' = \text{red}^{-1}(\bar{x}')$ is identified with the extension of scalars of $A = \text{red}^{-1}(\bar{\pi}(\bar{x}'))$. It is easy to see that $\pi : A' \rightarrow A$ takes the skeleton of A' isomorphically onto the skeleton of A . \blacksquare

5.11. Extending morphisms to semistable models: analytic criteria. Let X be a smooth, proper, connected K -curve and let \mathfrak{X} be a semistable formal model of X . The *inverse image topology* on X^{an} is the topology $\mathcal{T}(\mathfrak{X})$ whose open sets are the sets of the form $\text{red}^{-1}(\mathfrak{U}_k)$, where \mathfrak{U}_k is a Zariski-open subset of \mathfrak{X}_k . Note that any such set is *closed* in the natural topology on X^{an} . If $\mathfrak{U}_k \subset \mathfrak{X}_k$ is an affine open subset, then \mathfrak{U}_k is the special fiber of a formal affine open $\mathfrak{U} = \text{Spf}(\mathcal{A}) \subset \mathfrak{X}$, the set underlying the generic fiber $\mathfrak{U}_K = \mathcal{M}(\mathcal{A}_K) \subset X^{\text{an}}$ is equal to $\text{red}^{-1}(\mathfrak{U}_k)$, and \mathcal{A} is equal to the ring $\hat{\mathcal{A}}_K$ of power-bounded elements in the affinoid algebra \mathcal{A}_K . This essentially means that \mathfrak{X} is a *formal analytic variety* in the sense of [BL85]; see [BPR11, Remark 5.30(3)] for an explanation. Therefore \mathfrak{X} is constructed by gluing the canonical models of the affinoid subdomains of X^{an} which are open in the inverse image topology $\mathcal{T}(\mathfrak{X})$, along the canonical models of their intersections; here the *canonical model* of an affinoid space $\mathcal{M}(\mathcal{B})$ is by definition $\text{Spf}(\hat{\mathcal{B}})$. The following general fact about formal analytic varieties follows from these observations and the functoriality of the reduction map.

Proposition 5.12. *Let X, X' be smooth, proper, connected algebraic curves over K , let $\varphi : X' \rightarrow X$ be a finite morphism, and let \mathfrak{X} and \mathfrak{X}' be semistable formal models of X and X' , respectively. Then φ extends to a morphism $\mathfrak{X}' \rightarrow \mathfrak{X}$ if and only if φ is continuous with respect to the inverse image topologies $\mathcal{T}(\mathfrak{X})$ and $\mathcal{T}(\mathfrak{X}')$, in which case there is exactly one morphism $\mathfrak{X}' \rightarrow \mathfrak{X}$ extending φ .*

Fix smooth, proper, connected algebraic K -curves X, X' and a finite morphism $\varphi : X' \rightarrow X$. The following theorem is well-known to experts, although no proof appears in the literature to the best of our knowledge.

Theorem 5.13. *Let \mathfrak{X} and \mathfrak{X}' be semistable formal models of X and X' , respectively. Then $\varphi : X' \rightarrow X$ extends to a morphism $\mathfrak{X}' \rightarrow \mathfrak{X}$ if and only if $\varphi^{-1}(V(\mathfrak{X})) \subset V(\mathfrak{X}')$, and $\mathfrak{X}' \rightarrow \mathfrak{X}$ is finite if and only if $\varphi^{-1}(V(\mathfrak{X})) = V(\mathfrak{X}')$.*

Remark 5.14. If \mathfrak{X} and \mathfrak{X}' are semistable formal models of the punctured curves (X, D) and (X', D') , respectively, and $\varphi : X' \rightarrow X$ is a finite morphism with $\varphi^{-1}(D) = D'$ which extends to a morphism $\mathfrak{X}' \rightarrow \mathfrak{X}$, then since $\varphi^{-1}(V(\mathfrak{X})) \subset V(\mathfrak{X}')$ it follows from Remark 4.29 that there is a natural harmonic morphism of metric graphs $\Sigma(X', V(\mathfrak{X}') \cup D') \rightarrow \Sigma(X, V(\mathfrak{X}) \cup D)$. The morphism $\mathfrak{X}' \rightarrow \mathfrak{X}$ is finite if and only if the local degree of this harmonic morphism at every $v' \in V(\mathfrak{X}')$ is positive.

Before giving a proof of Theorem 5.13, we mention the following consequences. Let \mathfrak{X}_1 and \mathfrak{X}_2 be semistable formal models of X . We say that \mathfrak{X}_1 *dominates* \mathfrak{X}_2 provided that there exists a (necessarily unique) morphism $\mathfrak{X}_1 \rightarrow \mathfrak{X}_2$ inducing the identity map on analytic generic fibers. Taking $X = X'$ in Theorem 5.13, we obtain the following corollary; this gives a different proof of the second part of [BPR11, Theorem 5.38].

Corollary 5.15. *Let \mathfrak{X}_1 and \mathfrak{X}_2 be semistable formal models of X . Then \mathfrak{X}_1 dominates \mathfrak{X}_2 if and only if $V(\mathfrak{X}_1) \supset V(\mathfrak{X}_2)$.*

In conjunction with Proposition 5.10, we obtain the following corollary. If K' is a complete and algebraically closed field extension of K , we say that a semistable formal model \mathfrak{X}' of $X_{K'}$ is *defined over R* provided that it arises as the extension of scalars of a (necessarily unique) semistable formal model of X .

Corollary 5.16. *Let K' be a complete and algebraically closed field extension of K , let \mathfrak{X}' be a semistable formal model of $X_{K'}$, and suppose that there exists a semistable formal model \mathfrak{X} of X such that $\mathfrak{X}_{R'}$ dominates \mathfrak{X}' . Then \mathfrak{X}' is defined over R .*

Proof. Let $\pi : X_{K'}^{\text{an}} \rightarrow X^{\text{an}}$ be the canonical map. By Corollary 5.15 we have $V(\mathfrak{X}_{R'}) \supset V(\mathfrak{X}')$, and by Proposition 5.10 the map π defines a bijection $V(\mathfrak{X}_{R'}) \xrightarrow{\sim} V(\mathfrak{X})$ and an isomorphism

$$\Sigma(X_{K'}, V(\mathfrak{X}_{R'})) \xrightarrow{\sim} \Sigma(X, V(\mathfrak{X})).$$

Let $V = V(\mathfrak{X})$ and $V' = \pi(V(\mathfrak{X}'))$. We claim that V' is a semistable vertex set. Granted this, letting \mathfrak{Y} be the semistable formal model of X associated to V' , we have $V(\mathfrak{Y}_{R'}) = V(\mathfrak{X}')$ since $V(\mathfrak{Y}_{R'}) \subset V(\mathfrak{X}_{R'})$ (as $\mathfrak{X}_{R'}$ dominates $\mathfrak{Y}_{R'}$) and $\pi(V(\mathfrak{Y}_{R'})) = V(\mathfrak{Y})$, so $\mathfrak{Y}_{R'} = \mathfrak{X}'$.

It remains to prove the claim that V' is a semistable vertex set. The point is that the question of whether or not a subset of V is a semistable vertex set is intrinsic to the augmented Λ -metric graph $\Sigma = \Sigma(X, V) \cong \Sigma(X_{K'}, V(\mathfrak{X}_{R'}))$. We leave the details to the reader. ■

We will use the following lemmas in the proof of Theorem 5.13. Recall from (3.11) that if V is a semistable vertex set of X , then a connected component C of $X^{\text{an}} \setminus V$ is *adjacent* to a point $x \in V$ provided that the closure of C in X^{an} contains x .

Lemma 5.17. *Let \mathfrak{X} be a semistable formal model of X , let $V = V(\mathfrak{X})$, let $\mathfrak{U}_k \subset \mathfrak{X}_k$ be a subset, and let $U = \text{red}^{-1}(\mathfrak{U}_k) \subset X^{\text{an}}$. Then U is open in the topology $\mathcal{T}(\mathfrak{X})$ if and only if the following conditions hold:*

- (1) U is closed in the ordinary topology on X^{an} , and
- (2) for every $x \in U \cap V$, all but finitely many connected components of $X^{\text{an}} \setminus V$ which are adjacent to x are contained in U .

Proof. Let $\bar{\zeta}$ be a generic point of \mathfrak{X}_k and let $\bar{y} \in \mathfrak{X}_k$ be a closed point. Let $x \in V$ be the unique point of X^{an} reducing to $\bar{\zeta}$ and let $B = \text{red}^{-1}(\bar{y})$. By the anti-continuity of the reduction map, \bar{y} is in the closure of $\{\bar{\zeta}\}$ if and only if x is adjacent to B . The lemma follows easily from this and the fact that the connected components of $X^{\text{an}} \setminus V$ are exactly the inverse images of the closed points of \mathfrak{X}_k under red . ■

Lemma 5.18. *Let V and V' be semistable vertex sets of X and X' , respectively, and suppose that $\varphi^{-1}(V) \subset V'$. Let C' be a connected component of $X'^{\text{an}} \setminus \varphi^{-1}(V)$. Then C' has the following form:*

- (1) If C' intersects $\Sigma(X', V')$, then $C' = \tau^{-1}(\Sigma(X', V') \cap C')$, and
- (2) otherwise C' is an open ball connected component of $X'^{\text{an}} \setminus V'$.

Proof. Let $\Sigma' = \Sigma(X', V')$. Suppose that there exists $y' \in C'$ such that $\tau(y') \notin C'$. Let B' be the connected component of $X'^{\text{an}} \setminus \Sigma'$ containing y' , so B' is an open ball contained in C' and $\tau(y')$ is the end of B' . It follows that $\tau(y')$ is contained in the closure of C' in X'^{an} . Since C' is a connected component of $X'^{\text{an}} \setminus \varphi^{-1}(V)$, its closure is contained in $C' \cup \varphi^{-1}(V)$, so $\tau(y') \in \varphi^{-1}(V)$. Therefore $B' = (B' \cup \{\tau(y')\}) \cap (X'^{\text{an}} \setminus \varphi^{-1}(V))$ is open and closed in $X'^{\text{an}} \setminus \varphi^{-1}(V)$, so B' is a connected component of $X'^{\text{an}} \setminus \varphi^{-1}(V)$ and hence $B' = C'$.

Now suppose that $\tau(C') \subset C'$. Then $\tau(C') \subset C' \cap \Sigma'$, so $C' \subset \tau^{-1}(C' \cap \Sigma')$. Let $y' \in \tau^{-1}(C' \cap \Sigma')$ and let B' be the connected component of $X'^{\text{an}} \setminus \Sigma'$ containing y' , as above. Since $\bar{B}' = B' \cup \{\tau(y')\}$ is a connected subset of $X'^{\text{an}} \setminus \varphi^{-1}(V)$ intersecting C' we have $\bar{B}' \subset C'$, so $y' \in C'$ and therefore $C' = \tau^{-1}(C' \cap \Sigma')$. ■

Proof of Theorem 5.13. Let $V = V(\mathfrak{X})$ and $V' = V(\mathfrak{X}')$. If there is an extension $\mathfrak{X}' \rightarrow \mathfrak{X}$ of φ , then the square

$$(5.18.1) \quad \begin{array}{ccc} X'^{\text{an}} & \xrightarrow{\varphi} & X^{\text{an}} \\ \text{red} \downarrow & & \downarrow \text{red} \\ \mathfrak{X}'_k & \longrightarrow & \mathfrak{X}_k \end{array}$$

commutes. Let $x' \in \varphi^{-1}(V)$, so $x = \varphi(x')$ reduces to a generic point $\bar{\zeta}$ of \mathfrak{X}_k . Since the reduction $\bar{\zeta}'$ of x' maps to $\bar{\zeta}$, the point $\bar{\zeta}'$ is generic, so $x' \in V'$. Therefore $\varphi^{-1}(V) \subset V'$. The morphism $\mathfrak{X}' \rightarrow \mathfrak{X}$ is finite if and only if every generic point of \mathfrak{X}'_k maps to a generic point of \mathfrak{X}_k ; as above, this is equivalent to $V' = \varphi^{-1}(V)$.

It remains to prove that if $\varphi^{-1}(V) \subset V'$, then φ extends to a morphism $\mathfrak{X}' \rightarrow \mathfrak{X}$. By Proposition 5.12, we must show that φ is continuous with respect to the topologies $\mathcal{T}(\mathfrak{X})$ and $\mathcal{T}(\mathfrak{X}')$. Let $U \subset X^{\text{an}}$ be $\mathcal{T}(\mathfrak{X})$ -open and let $U' = \varphi^{-1}(U)$. Clearly U is closed, so U' is closed (with respect to the ordinary topologies). We must show that condition (2) of Lemma 5.17 holds for U' . Let $x' \in U' \cap V'$ and let $x = \varphi(x')$. If $x \notin V$, then let $C \subset U$ be the connected component of $X^{\text{an}} \setminus V$ containing x and let C' be the connected component of $X'^{\text{an}} \setminus \varphi^{-1}(V)$ containing x' . Then $C' \subset U'$ because $\varphi(C') \subset C$, and C' contains every connected component of $X'^{\text{an}} \setminus V'$ adjacent to x' since C' is an open neighborhood of x' .

Now suppose that $x \in V$. Any connected component of $X'^{\text{an}} \setminus V'$ which is adjacent to x' maps into a connected component of $X^{\text{an}} \setminus V$ which is adjacent to x . There are finitely many connected components $X^{\text{an}} \setminus V$ which are adjacent to x and not contained in U by Lemma 5.17. Let C be such a component. Since φ is finite, there are only finitely many connected components of $\varphi^{-1}(C)$; each of these is a connected component of $X'^{\text{an}} \setminus \varphi^{-1}(V)$. If C' is such a connected component, then either C' is an open ball connected component of $X'^{\text{an}} \setminus V'$ or C' intersects $\Sigma' = \Sigma(X', V')$ by Lemma 5.18. There are finitely many connected components of $X'^{\text{an}} \setminus V'$ which intersect Σ' — these are just the open annulus connected components of $X'^{\text{an}} \setminus V'$ — so there are only finitely many connected components of $X'^{\text{an}} \setminus V'$ contained in $\varphi^{-1}(C)$. Therefore all but finitely many connected components of $X'^{\text{an}} \setminus V'$ which are adjacent to x' map to connected components of $X^{\text{an}} \setminus V$ which are contained in U . ■

5.19. Simultaneous semistable reduction theorems. Recall that K_0 is a field equipped with a non-trivial non-Archimedean valuation $\text{val} : K_0 \rightarrow \mathbf{R} \cup \{\infty\}$. As above we fix an algebraic closure \bar{K}_0 of K_0 and a valuation val on \bar{K}_0 extending the given valuation on K_0 ; for any field $K_1 \subset \bar{K}_0$ we consider K_1 as a valued field with respect to the restriction of val , and we write R_1 for the valuation ring of K_1 . In what follows, X and X' are smooth, proper, geometrically connected K_0 -curves and $\varphi : X' \rightarrow X$ is a finite morphism.

Proposition 5.20. *Let \mathcal{X} be a semistable R_0 -model of X . If there exists a semistable R_0 -model \mathcal{X}' of X' such that $\varphi : X' \rightarrow X$ extends to a finite morphism $\mathcal{X}' \rightarrow \mathcal{X}$, then there is exactly one such model \mathcal{X}' up to (unique) isomorphism.*

Proof. Suppose first that $K = K_0$ is complete and algebraically closed. Let \mathfrak{X} (resp. \mathfrak{X}') be the ϖ -adic completion of \mathcal{X} (resp. \mathcal{X}'). In this case, the proposition follows from Theorem 5.8 and the fact that if the morphism $\mathfrak{X}' \rightarrow \mathfrak{X}$ of Theorem 5.13 is finite, then $V(\mathfrak{X}')$ is uniquely determined by $V(\mathfrak{X})$. The general case follows from this case after passing to the completion of the algebraic closure K of K_0 : if $\mathcal{X}', \mathcal{X}''$ are two semistable models of X' such that φ extends to finite morphisms $\mathcal{X}' \rightarrow \mathcal{X}$ and $\mathcal{X}'' \rightarrow \mathcal{X}$, then the isomorphism $\mathcal{X}''_R \xrightarrow{\sim} \mathcal{X}'_R$ descends to an isomorphism $\mathcal{X}'' \xrightarrow{\sim} \mathcal{X}'$ by Lemma 5.5. ■

Let $\mathcal{X}_1, \mathcal{X}_2$ be semistable R_0 -models of X . Recall that \mathcal{X}_1 *dominates* \mathcal{X}_2 if there exists a (necessarily unique) morphism $\mathcal{X}_1 \rightarrow \mathcal{X}_2$ inducing the identity on X .

Proposition 5.21. *Let \mathcal{X} and \mathcal{X}' be semistable R_0 -models of X and X' , respectively. Let $D' \subset X'(K_0)$ be a finite set of points. Then there exists a finite, separable extension K_1 of K_0 and a semistable R_1 -model \mathcal{X}'' of (X'_{K_1}, D') such that:*

- (1) \mathcal{X}'' dominates \mathcal{X}'_{R_1} ,
- (2) $\varphi_{K_1} : X'_{K_1} \rightarrow X_{K_1}$ extends to a morphism $\mathcal{X}'' \rightarrow \mathcal{X}_{R_1}$, and
- (3) any other semistable R_1 -model of (X'_{K_1}, D') satisfying the above two properties dominates \mathcal{X}'' .

Moreover, the formation of \mathcal{X}'' commutes with arbitrary valued field extensions $K_1 \rightarrow K'_1$.

Proof. Suppose first that $K_0 = K$ is complete and algebraically closed. Let \mathfrak{X} and \mathfrak{X}' be the ϖ -adic completions of \mathcal{X} and \mathcal{X}' , respectively. Let $V = V(\mathfrak{X})$ and $V' = V(\mathfrak{X}')$. By Lemma 3.15, there is a minimal semistable vertex set V'' of (X', D') which contains $\varphi^{-1}(V) \cup V'$. Let \mathfrak{X}'' be the semistable formal model of X' corresponding to V'' . Then \mathfrak{X}'' dominates \mathfrak{X}' by Corollary 5.15 and φ extends to a morphism $\mathfrak{X}'' \rightarrow \mathfrak{X}$ by Theorem 5.13. Part (3) follows from Corollary 5.15 and the minimality of V'' . Taking \mathcal{X}'' to be the algebraization of \mathfrak{X}'' yields (1)–(3) in this case.

For a general valued field K_0 , suppose that K is the completion of \overline{K}_0 . By Lemma 5.4 the model of X'_{K_0} constructed above descends to a model \mathcal{X}'' defined over the ring of integers of a finite, separable extension K_1 of K_0 . Properties (1)–(3) follow from Lemma 5.5 and the corresponding properties of $\mathcal{X}''_{R'}$.

Now we address the behavior of this construction with respect to base change. Using Lemma 5.5 we immediately reduce to the case of an extension $K \rightarrow K'$ of complete and algebraically closed valued fields. Let R' be the ring of integers of K' . Let \mathcal{X}'' (resp. \mathcal{X}''') be the minimal R -model of X' dominating \mathcal{X}' (resp. R' -model of $X'_{K'}$ dominating $\mathcal{X}'_{R'}$) mapping to \mathcal{X} (resp. $\mathcal{X}_{R'}$). By (3) as applied to \mathcal{X}''' , we have that $\mathcal{X}''_{R'}$ dominates \mathcal{X}''' . By Corollary 5.16, \mathcal{X}''' is defined over R , so by (3) as applied to \mathcal{X}'' , we have $\mathcal{X}''_{R'} = \mathcal{X}'''$. \blacksquare

Theorem 5.22. (Liu) *Let \mathcal{X} (resp. \mathcal{X}') be a semistable R_0 -model of X (resp. X'). Let $D \subset X(K_0)$ and $D' \subset X'(K_0)$ be finite sets, and suppose that $\varphi(D') \subset D$. Then there exists a finite, separable extension K_1 of K_0 and semistable R_1 -models $\mathcal{X}_1, \mathcal{X}'_1$ of (X_{K_1}, D) , (X'_{K_1}, D') , respectively, such that*

- (1) \mathcal{X}_1 dominates \mathcal{X}_{R_1} and \mathcal{X}'_1 dominates \mathcal{X}'_{R_1} ,
- (2) the morphism $\varphi_{K_1} : X'_{K_1} \rightarrow X_{K_1}$ extends to a finite morphism $\mathcal{X}'_1 \rightarrow \mathcal{X}_1$, and
- (3) if $\mathcal{X}_2, \mathcal{X}'_2$ are semistable formal models of (X_{K_1}, D) , (X'_{K_1}, D') , respectively, satisfying (1) and (2) above, then \mathcal{X}_2 dominates \mathcal{X}_1 and \mathcal{X}'_2 dominates \mathcal{X}'_1 .

Moreover, the formation of $\mathcal{X}'_1 \rightarrow \mathcal{X}_1$ commutes with arbitrary valued field extensions $K_1 \rightarrow K'_1$.

The morphism $\mathcal{X}'_1 \rightarrow \mathcal{X}_1$ is called the *stable marked hull* of $\mathcal{X}' \dashrightarrow \mathcal{X}$ in [Liu06].

Proof. First assume that $K_0 = K$ is complete and algebraically closed; let $\mathfrak{X}, \mathfrak{X}'$ be the ϖ -adic completions of $\mathcal{X}, \mathcal{X}'$, respectively. Let $V = V(\mathfrak{X})$ and $V' = V(\mathfrak{X}')$. By Theorem 5.8, Theorem 5.13, and Corollary 5.15, we may equivalently formulate the existence and uniqueness of $\mathfrak{X}'_1 \rightarrow \mathfrak{X}_1$ in terms of semistable vertex sets, as follows. We must prove that there exists a semistable vertex set V_1 of (X, D) such that

- (1) V_1 contains $V \cup \varphi(V')$,
- (2) $\varphi^{-1}(V_1)$ is a semistable vertex set of (X', D') , and
- (3) V_1 is minimal in the sense that if V_2 is another semistable vertex set of (X, D) satisfying (1) and (2) above, then $V_2 \supset V_1$.

First we will prove the existence of V_1 satisfying (1) and (2). By Lemma 3.15 we may enlarge V to assume that V is a semistable vertex set of (X, D) . Let $\Sigma = \Sigma(X, V \cup D)$ and $\Sigma' = \Sigma(X', V')$. By Corollary 4.18, there exists a skeleton Σ_1 of (X, D) such that $\Sigma_1 \supset \Sigma \cup \varphi(\Sigma')$ and such that $\Sigma'_1 = \varphi^{-1}(\Sigma_1)$ is a skeleton of $(X', \varphi^{-1}(D))$. Let V'_1 be a vertex set for Σ'_1 . Since any finite subset of type-2 points of Σ'_1 which contains V'_1 is again a vertex set for Σ'_1 by Proposition 3.12(4), we may and do assume that $V' \subset V'_1$. Let $V_1 \subset \Sigma_1$ be the union of a vertex set for Σ_1 with $V \cup \varphi(V'_1)$. Then V_1 is

a semistable vertex set of (X, D) , and $\varphi^{-1}(V_1) \subset \Sigma'_1$ is a finite set of type-2 points containing V'_1 , thus is a semistable vertex set of $(X', \varphi^{-1}(D))$ (hence of (X', D') as well).

To prove that there exists a minimal such V_1 , we make the following recursive construction. Let $V(0) = V$, let $V'(0) = V'$, and for each $n \geq 1$ let $V(n)$ be the minimal semistable vertex set of (X, D) containing $V(n-1) \cup \varphi(V'(n-1))$ and let $V'(n)$ be the minimal semistable vertex set of (X', D') containing $\varphi^{-1}(V(n))$. These sets exist by Lemma 3.15. By induction it is clear that if V_2 is any semistable vertex set of (X, D) satisfying (1) and (2) above, then $V(n) \subset V_2$ for each n . Since V_2 is a finite set, for some n we have $V(n) = V(n+1)$, which is to say that $\varphi(V'(n)) \subset V(n)$; since $V'(n) \supset \varphi^{-1}(V(n))$, we have that $V'(n) = \varphi^{-1}(V(n))$ is a semistable vertex set of (X', D') . Then $V_1 = V(n)$ is the minimal semistable vertex set satisfying (1) and (2) above.

The case of a general ground field reduces to the geometric case handled above exactly as in the proof of Proposition 5.21, as does the statement about the behavior of the stable marked hull with respect to valued field extensions. \blacksquare

Remark 5.23. (The skeletal viewpoint on Liu's theorem) The statement of Theorem 5.22 is strongly analogous to Corollary 4.18, which is indeed the main ingredient in the proof. The difference is that whereas finite morphisms of semistable models correspond to pairs V, V' of semistable vertex sets such that $\varphi^{-1}(V) = V'$, for finite morphisms of triangulated punctured curves one requires in addition that $\varphi^{-1}(\Sigma) = \Sigma'$. The former condition (of Liu's theorem) does not imply the latter: for instance, let X' be the Tate curve $y^2 = x^3 - x + \varpi$, let $X = \mathbf{P}^1$, and let $\varphi : X' \rightarrow X$ be the cover $(x, y) \mapsto x$. This extends to a finite morphism of semistable models given by the same equations; however, the associated skeleton of X' (resp. of \mathbf{P}^1) is a circle (resp. a point), so $\varphi^{-1}(\Sigma) \subsetneq \Sigma'$.

In general, if $\varphi^{-1}(V) = V'$ and $\varphi^{-1}(D) = D'$, then by Remark 4.19 the image of the skeleton $\Sigma(X', V' \cup D')$ is equal to $\Sigma(X, V \cup D)$ union a finite number of geodesic segments T_1, \dots, T_r , and by Theorem 4.13 the saturation $\varphi^{-1}(\varphi(\Sigma(X', V' \cup D')))$ is a skeleton of X' .

Theorem 5.25 below follows the same philosophy in deriving a simultaneous semistable reduction theorem of Liu-Lorenzini from Proposition 4.12.

Remark 5.24. Liu in fact works over an arbitrary Dedekind scheme (a connected Noetherian regular scheme of dimension 1), which includes discrete valuation rings but not more general valuation rings. In his statement of Theorem 5.22 the given models $\mathcal{X}, \mathcal{X}'$ are allowed to be any integral, projective $\text{Spec}(R_0)$ -schemes with generic fibers X and X' , respectively. Although we restrict to semistable models $\mathcal{X}, \mathcal{X}'$ in this paper, using a more general notion of a triangulation our methods can be extended to treat relatively normal models $\mathcal{X}, \mathcal{X}'$.

The following simultaneous stable reduction theorem can be found in [LL99]. Theorem 5.25 is to Proposition 4.12 as Theorem 5.22 is to Corollary 4.18

Theorem 5.25. (Liu-Lorenzini) *Suppose that (X, D) and (X', D') are both stable and that $\varphi^{-1}(D) = D'$. Assume that (X, D) and (X', D') admit stable models \mathcal{X} and \mathcal{X}' , respectively, defined over R_0 . Then $\varphi : X' \rightarrow X$ extends to a (not necessarily finite) morphism $\mathcal{X}' \rightarrow \mathcal{X}$.*

Proof. By Lemma 5.5 we may assume that $K_0 = K$ is complete and algebraically closed. Let V (resp. V') be the minimal semistable vertex set of (X, D) (resp. (X', D')). By Theorem 5.13, we must show that $\varphi^{-1}(V) \subset V'$. Let $\Sigma = \Sigma(X, V)$ and $\Sigma' = \Sigma(X', V')$. Let V_1 be a vertex set for Σ with respect to which Σ has no loop edges. Recall that

$$V = \{x \in V_1 : g(x) \geq 1 \text{ or } x \text{ has valency at least } 3\}$$

by Proposition 3.18(2). Let $V'_1 = \varphi^{-1}(V_1) \cup V'$; this is a vertex set for Σ' since $\varphi^{-1}(\Sigma) \subset \Sigma'$ by Proposition 4.12(2). If $x \in V$ has genus at least 1, then any $x' \in \varphi^{-1}(x)$ has genus at least 1, so $\varphi^{-1}(x) \subset V'$. Let $x \in V$ have genus 0, so x has valency at least 3 in Σ . Let $x' \in \varphi^{-1}(x) \subset \Sigma'$ and let U' be the set containing x' and all of the connected components of $X'^{\text{an}} \setminus V'_1$ adjacent to x' . Then U' is an open neighborhood of x' , so $\varphi(U')$ is an open neighborhood of x by [Ber90, Lemma 3.2.4]. Let A be an open annulus connected component of $X^{\text{an}} \setminus V_1$ adjacent to x . Any open neighborhood of x intersects A , so there exists $y' \in U'$ with $\varphi(y') \in A$. Let A' be the connected component of $U' \setminus \{x'\}$

containing y' , so $\varphi(A') \subset A$. Since x' is an end of A' and $\varphi(x')$ is an end of A , Lemma 4.3 implies that A' is an open annulus. Since x has valency at least 3 in Σ (and since Σ has no loop edges), there are at least 3 distinct open annulus connected components of $X^{\text{an}} \setminus V_1$ adjacent to x . By the above, the same is true of x' , so x' has valency at least 3 in Σ' , so $x' \in V'$, as desired. \blacksquare

Remark 5.26. In Theorem 5.25 it is not enough to assume that $\varphi(D') \subset D$. For instance, let $X = X' = \mathbf{P}^1$, let $\varphi : X' \rightarrow X$ be the identity, let $D = \{0, 1, \infty\}$, and let $D' = \{0, \varpi, \infty\}$, where $\varpi \in \mathfrak{m}_R \setminus \{0\}$. The minimal semistable vertex set of (X, D) is the maximal point of $\mathbf{B}(1)$ and the minimal semistable vertex set of (X', D') is the maximal point of $\mathbf{B}(\varpi)$, so φ does not extend to a morphism of stable models.

6. A LOCAL LIFTING THEOREM

6.1. Let X be a smooth, proper, connected curve over K and let $x \in X^{\text{an}}$ be a type-2 point. Let $V \subset X^{\text{an}}$ be a strongly semistable vertex set of X containing x , let $\Sigma = \Sigma(X, V)$, and let $\tau : X^{\text{an}} \rightarrow \Sigma$ be the retraction. Let e_1, \dots, e_r be the edges of Σ adjacent to x and let $\Sigma_0 = \{x\} \cup e_1^\circ \cup \dots \cup e_r^\circ$. Then Σ_0 is an open neighborhood of x in Σ and $\tau^{-1}(\Sigma_0)$ is an open neighborhood of x in X^{an} . Following [BPR11, 5.54] and [Ber93], we define a *simple neighborhood* of x to be an open neighborhood of this form (for some choice of V). The connected components of $\tau^{-1}(\Sigma_0) \setminus \{x\}$ are open balls and the open annuli $\tau^{-1}(e_1^\circ), \dots, \tau^{-1}(e_r^\circ)$.

Definition 6.2. A *star-shaped curve* is a pointed K -analytic space (Y, y) which is isomorphic to (U, x) where x is a type-2 point in the analytification of a smooth, proper, connected curve over K and U is a simple neighborhood of x . The point y is called the *central vertex* of Y .

6.3. Let (Y, y) be a star-shaped curve, so $Y \setminus \{y\}$ is a disjoint union of open balls and finitely many open annuli A_1, \dots, A_r . The *skeleton* of Y is defined to be the set $\Sigma(Y, \{y\}) = \{y\} \cup \bigcup_{i=1}^r \Sigma(A_i)$. A *compatible divisor* in Y is a finite set $D \subset Y(K)$ whose points are contained in distinct open ball connected components of $Y \setminus \{y\}$, so the connected components of $Y \setminus (\{y\} \cup D)$ are open balls, the open annuli A_1, \dots, A_r , and (finitely many) open balls B_1, \dots, B_s punctured at a point of D . The data (Y, y, D) of a star-shaped curve along with a compatible divisor is called a *punctured star-shaped curve*. The *skeleton* of (Y, y, D) is the set

$$\Sigma(Y, \{y\} \cup D) = \{y\} \cup D \cup \bigcup_{i=1}^r \Sigma(A_i) \cup \bigcup_{j=1}^s \Sigma(B_j).$$

Fix a compatible divisor D in Y , and let $\Sigma_0 = \Sigma(Y, \{y\} \cup D)$. There is a canonical continuous retraction map $\tau : Y \rightarrow \Sigma_0$ defined exactly as for skeleta of algebraic curves (3.11). The connected components of $\Sigma_0 \setminus \{y\}$ are called the *edges* of Σ_0 ; an edge is called *infinite* if it contains a point of D and *finite* otherwise.

If Y is the simple neighborhood $\tau^{-1}(\Sigma_0)$ of $x \in X^{\text{an}}$ as above, then $\Sigma(Y, \{x\}) = \Sigma_0 = \Sigma \cap Y$. A finite set $D \subset Y(K)$ is compatible with Y if and only if V is a semistable vertex set for (X, D) , in which case $\Sigma(Y, \{x\} \cup D) = Y \cap \Sigma(X, V \cup D)$. The retraction $\tau : Y \rightarrow \Sigma(Y, \{x\} \cup D)$ is the restriction of the canonical retraction $\tau : X^{\text{an}} \rightarrow \Sigma(X, V \cup D)$.

6.4. Let (Y, y) be a star-shaped curve. Then Y is proper as a K -analytic space if and only if all connected components of $Y \setminus \{y\}$ are open balls, i.e. if and only if $\Sigma(Y, \{y\}) = \{y\}$. If Y is proper, then there is a smooth, proper, connected curve X over K and an isomorphism $f : Y \xrightarrow{\sim} X^{\text{an}}$. Let $x = f(y)$. Then $\{x\}$ is a semistable vertex set of X , so by Theorem 5.8 there is a unique smooth formal model \mathfrak{X} of X^{an} such that x reduces to the generic point of \mathfrak{X}_k . Let $D \subset Y(K)$ be a finite set. Then D is compatible with Y if and only if the points of $f(D)$ reduce to distinct closed points of \mathfrak{X}_k .

Conversely, let \mathfrak{X} be a smooth, proper, connected formal curve over $\text{Spf}(R)$. If $x \in X^{\text{an}}$ is the point reducing to the generic point of \mathfrak{X}_k , then (\mathfrak{X}_K, x) is a proper star-shaped curve.

Proposition 6.5. *Let (Y, y) be a star-shaped curve. Then (Y, y) is isomorphic to (U, x) where x is the central vertex of a proper star-shaped curve X and U is a simple neighborhood of x .*

Proof. Let A_1, \dots, A_r be the open annulus connected components of $Y \setminus \{y\}$. Choose isomorphisms $f_i : A_i \xrightarrow{\sim} \mathbf{S}(a_i)_+$ with standard open annuli such that $f_i(x)$ approaches the Gauss point of $\mathbf{B}(1)$ as x approaches y . Let X be the curve obtained from Y by gluing an open ball $\mathbf{B}(1)_+$ onto each A_i via the inclusions $f_i : A_i \xrightarrow{\sim} \mathbf{S}(a_i)_+ \subset \mathbf{B}(1)_+$. Then X is a smooth, proper, connected K -analytic curve, and it is clear from the construction that (X, y) is star-shaped and that Y is a simple neighborhood of y in X . \blacksquare

6.6. A proper star-shaped curve X and an inclusion $i : Y \xrightarrow{\sim} U \subset X$ as in Proposition 6.5 is called a *compactification* of Y (as a star-shaped curve). Note that $X \setminus i(Y)$ is a disjoint union of finitely many closed balls, one for each open annulus connected component of $Y \setminus \{y\}$.

6.7. Let (Y, y) be a star-shaped curve. The smooth, proper, connected k -curve C_y with function field $\widetilde{\mathcal{H}}(y)$ is called the *residue curve* of Y . The tangent vectors in T_y are naturally in bijective correspondence with the connected components of $Y \setminus \{y\}$. We define a *reduction map* $\text{red} : Y \rightarrow C_y$ by sending y to the generic point of C_y , and sending every point in a connected component B of $Y \setminus \{y\}$ to the closed point of C_y corresponding to the tangent vector determined by B . This sets up a one-to-one correspondence between the connected components of $Y \setminus \{y\}$ and the closed points of C_y .

Remark 6.8.

- (1) When Y is proper, so $Y \cong \mathfrak{X}_K$ for a smooth, proper, connected formal curve over $\text{Spf}(R)$, then $C_y \cong \mathfrak{X}_k$ and the reduction map $Y \rightarrow C_y$ coincides with the canonical reduction map $\mathfrak{X}_K \rightarrow \mathfrak{X}_k$.
- (2) Let $D \subset Y(K)$ be a compatible divisor and let $\Sigma_0 = \Sigma(Y, \{y\} \cup D)$. Every edge e of Σ_0 is contained in a unique connected component of $Y \setminus \{y\}$, and $\text{red}(e) \in C_y(k)$ is the closed point corresponding to the tangent direction represented by e .

6.9. Tame coverings. We now study a class of morphisms of star-shaped curves analogous to (4.30). We begin with the following technical result.

Lemma 6.10. *Let A, A' be open annuli or punctured open balls and let $\varphi : A' \rightarrow A$ be a finite morphism of degree δ . Suppose that δ is prime to $\text{char}(k)$ if $\text{char}(k) > 0$. Fix an isomorphism $A \cong \mathbf{S}(a)_+$ (where we allow $a = 0$).*

- (1) *There is an isomorphism $A' \cong \mathbf{S}(a')_+$ such that the composition*

$$(6.10.1) \quad \mathbf{S}(a')_+ \cong A' \xrightarrow{\varphi} A \cong \mathbf{S}(a)_+$$

is $t \mapsto t^\delta$.

- (2) *There is an isomorphism $A' \cong \mathbf{S}(a')_+$ such that (6.10.1) extends to a morphism $\psi : \mathbf{B}(1)_+ \rightarrow \mathbf{B}(1)_+$ with $\psi^{-1}(0) = 0$.*
- (3) *If (6.10.1) extends to a morphism $\psi : \mathbf{B}(1)_+ \rightarrow \mathbf{B}(1)_+$ for a given isomorphism $A' \cong \mathbf{S}(a')_+$, then the extension is unique.*

Proof. Let u be a parameter on A' , i.e. an isomorphism $u : A' \xrightarrow{\sim} \mathbf{S}(a')_+ \subset \mathbf{B}(1)_+$ with a standard open annulus. By Lemma 4.2, φ restricts to an affine map on skeleta $\Sigma(A') \rightarrow \Sigma(A)$ with degree δ . If $B = \mathbf{S}(b, c)$ is a closed sub-annulus of $\mathbf{S}(a')_+$, then by [Thu05, Lemme 2.2.1], after potentially replacing u by u^{-1} , $\varphi^*(t)$ has the form $\alpha u^\delta(1 + g(u))$ on B , where $\alpha \in R^\times$ and $|g|_{\text{sup}} < 1$. Since δ is not divisible by $\text{char}(k)$ if $\text{char}(k) > 0$, the Taylor expansion for $\sqrt[\delta]{1+g}$ has coefficients contained in R , hence converges to a δ th root of $1 + g(u)$ on A . Choosing a δ th root of α as well and letting $u_B = u \sqrt[\delta]{\alpha} \sqrt[\delta]{1+g}$, we have that u_B is a parameter on B such that $\varphi^*(t) = u_B^\delta$. Hence for each such B there are exactly δ choices of a parameter u_B on B such that $\varphi^*(t) = u_B^\delta$; choosing a compatible set of such parameters for all B yields a parameter on A' satisfying (1).

Part (2) follows immediately from (1). A morphism from a K -analytic space X to $\mathbf{B}(1)_+$ is given by a unique analytic function f on X such that $|f(x)| < 1$ for all $x \in X$, so (3) follows from the fact that the restriction homomorphism $\mathcal{O}(\mathbf{B}(1)_+) \rightarrow \mathcal{O}(\mathbf{S}(a')_+)$ is injective. ■

6.11. Let A be an open annulus. Let δ be a positive integer, and assume that δ is prime to the characteristic of k if $\text{char}(k) > 0$. Let A' be an open annulus and let $\varphi : A' \rightarrow A$ be a finite morphism of degree δ . By Lemma 6.10, the group $\text{Aut}_A(A')$ is isomorphic to $\mathbf{Z}/\delta\mathbf{Z}$. Moreover, if A'' is another open annulus and $\varphi' : A'' \rightarrow A$ is a finite morphism of degree δ , then there exists an A -isomorphism $\psi : A' \xrightarrow{\sim} A''$; the induced isomorphism $\text{Aut}_A(A') \xrightarrow{\sim} \text{Aut}_A(A'')$ is independent of the choice of ψ since both groups are abelian. Therefore the group $\text{Aut}_A(\delta) := \text{Aut}_A(A') \cong \mathbf{Z}/\delta\mathbf{Z}$ is canonically determined by A and δ .

Let $\varphi : C' \rightarrow C$ be a finite morphism of smooth, proper, connected k -curves. Recall that φ is *tamely ramified* if either $\text{char}(k) = 0$ or the ramification degree of φ at every closed point of C' is prime to the characteristic of k .

Remark 6.12.

- (1) Let $\varphi : C' \rightarrow C$ be a finite morphism. Let D be the (finite) set of branch points, let $D' = \varphi^{-1}(D) \subset C'(k)$, and let $U = C \setminus D$ and $U' = C' \setminus D'$. Then φ is tamely ramified if and only if $\varphi|_{U'} : U' \rightarrow U$ is a tamely ramified cover of U over C relative to D in the sense of [SGA1, XIII.2.1.3]: see the discussion in §2.0 of *loc. cit.*
- (2) A tamely ramified morphism is generically étale.
- (3) The Riemann-Hurwitz formula applies to a tamely ramified morphism.

Definition 6.13. Let (Y, y, D) be a punctured star-shaped curve with skeleton $\Sigma_0 = \Sigma(Y, \{y\} \cup D)$ and let C_y be the residue curve of Y . A *tame covering* of (Y, y, D) consists of a punctured star-shaped curve (Y', y', D') and a finite morphism $\varphi : Y' \rightarrow Y$ satisfying the following properties:

- (1) $\varphi^{-1}(y) = \{y'\}$,
- (2) $D' = \varphi^{-1}(D)$, and
- (3) if $C_{y'}$ denotes the residue curve of Y' and $\varphi_{y'} : C_{y'} \rightarrow C_y$ is the morphism induced by φ , then $\varphi_{y'}$ is tamely ramified and is branched only over the points of C_y corresponding to tangent directions at y represented by edges in Σ_0 .

Remark 6.14. The degree of φ is equal to the degree of $\varphi_{y'}$.

Example 6.15. Let $\varphi : (X', V', D') \rightarrow (X, V, D)$ be a tame covering of triangulated punctured curves. Let $\Sigma = \Sigma(X, V \cup D)$ and $\Sigma' = \Sigma(X', V' \cup D') = \varphi^{-1}(\Sigma)$. Let $x \in V$ be a finite vertex of Σ , let e_1, \dots, e_r be the finite edges of Σ adjacent to x , let $\Sigma_0 = \{x\} \cup \bigcup_{i=1}^r e_i^\circ$, let $Y = \tau^{-1}(\Sigma_0)$, and let $D_0 = D \cap Y$. Then (Y, x, D_0) is a punctured star-shaped curve. Let Σ'_0 be a connected component of $\varphi^{-1}(\Sigma_0)$, let $x' \in \Sigma'_0$ be the unique inverse image of x , let $Y' = \tau^{-1}(\Sigma'_0)$, and let $D'_0 = D' \cap Y'$. Then (Y', x', D'_0) is also a punctured star-shaped curve, and φ restricts to a finite morphism $\varphi : Y' \rightarrow Y$. This is in fact a tame covering of punctured star-shaped curves by Remark 4.34 and Proposition 4.35.

Proposition 6.16. Let $\varphi : (Y', y', D') \rightarrow (Y, y, D)$ be a degree- δ tame covering of punctured star-shaped curves.³ Let $\Sigma_0 = \Sigma(Y, \{y\} \cup D)$, let $\Sigma'_0 = \Sigma(Y', \{y'\} \cup D')$, let C_y (resp. $C_{y'}$) be the residue curve of Y (resp. Y'), and let $\varphi_{y'} : C_{y'} \rightarrow C_y$ be the induced morphism.

- (1) Let B be a connected component of $Y \setminus \{y\}$ disjoint from Σ_0 . Then φ^{-1} is a disjoint union of δ open balls mapping isomorphically onto B .
- (2) Let B be a connected component of $Y \setminus \{y\}$ meeting D , and choose an isomorphism of B with $\mathbf{B}(1)_+$ which identifies the unique point of $B \cap D$ with 0. Let B' be a connected component of $\varphi^{-1}(B)$. Then $\varphi|_{B'} : B' \rightarrow B$ is a finite morphism, the degree of $\varphi|_{B'}$ is the ramification degree δ' of $\varphi_{y'}$ at $\text{red}(B')$, and there is an isomorphism $B' \cong \mathbf{B}(1)_+$ sending the unique point

³This δ need not be prime to $\text{char}(k)$.

of $B' \cap D'$ to 0 such that the composition

$$\mathbf{B}(1)_+ \cong B' \xrightarrow{\varphi} B \cong \mathbf{B}(1)_+$$

is $t \mapsto t^{\delta'}$.

- (3) Let A be an open annulus connected component of $Y \setminus \{y\}$, and choose an isomorphism $A \cong \mathbf{S}(a)_+$. Let A' be a connected component of $\varphi^{-1}(A)$. Then $\varphi|_{A'} : A' \rightarrow A$ is a finite morphism, the degree of $\varphi|_{A'}$ is the ramification degree δ' of $\varphi_{y'}$ at $\text{red}(A')$, and there is an isomorphism of A' with an open annulus $\mathbf{S}(a')$ such that the composition

$$\mathbf{S}(a')_+ \cong A' \xrightarrow{\varphi} A \cong \mathbf{S}(a)_+$$

is $t \mapsto t^{\delta'}$.

- (4) φ is étale over $Y \setminus D$.
 (5) $\varphi^{-1}(\Sigma_0) = \Sigma'_0$.

Proof. In the situation of (1), let $\bar{x} = \text{red}(B)$. Since $\varphi_{y'}$ is not branched over \bar{x} , there are δ distinct points of $C_{y'}$ mapping to \bar{x} ; hence there are δ connected components B'_1, \dots, B'_δ of $Y' \setminus \{y'\}$ mapping onto B . The restriction of φ to each B'_i is finite of degree 1.

Next we prove (2). It is clear that $\varphi|_{B'} : B' \rightarrow B$ is finite, hence surjective; since B' is a connected component of $Y' \setminus \{y'\}$ containing a point of $D' = \varphi^{-1}(D)$, it is an open ball. Let x (resp. x') be the unique point of D (resp. D') contained in B (resp. B') and let $e \subset \Sigma_0$ (resp. $e' \subset \Sigma'_0$) be the edge adjacent to x (resp. x'). Then $B' \setminus \{x'\} \rightarrow B \setminus \{x\}$ is a finite morphism of punctured open balls, so by Lemma 3.3, Proposition 4.8, and Theorem 4.23(2), the restriction of φ to e' is affine morphism $e' \rightarrow e$ of degree δ' , and δ' is the degree of $B' \rightarrow B$. So by Lemma 6.10 there is an isomorphism $B' \cong \mathbf{B}(1)_+$ as described in the statement of the Theorem.

In the situation of (3), we claim that A' is an open annulus. Clearly $\varphi|_{A'} : A' \rightarrow A$ is finite, hence surjective. If A' is not an open annulus, then $A' \cong \mathbf{B}(1)_+$ is an open ball. The morphism $\mathbf{B}(1)_+ \cong A' \rightarrow A \cong \mathbf{S}(a)_+$ is given by a unit on $\mathbf{B}(1)_+$, which has constant absolute value; this contradicts surjectivity, so A' is in fact an open annulus. The proof now proceeds exactly as above.

Parts (4) and (5) follow immediately from parts (1)–(3). \blacksquare

Corollary 6.17. Let $\varphi : (Y', y', D'_0) \rightarrow (Y, y, D_0)$ be a tame covering of punctured star-shaped curves and let $i : (Y, y) \hookrightarrow (X, x)$ be a compactification of Y . Let D_1 be the union of $i(D_0)$ with a choice of K -point from every connected component of $X \setminus i(Y)$. Then there exists a compactification $i' : (Y', y') \hookrightarrow (X', x')$ and a tame covering $\psi : (X', x', D'_1) \rightarrow (X, x, D_1)$ such that $\psi \circ i' = i \circ \varphi$.

Proof. We compactify (Y', y') as in the proof of Proposition 6.5, gluing balls onto the annulus connected components of $Y' \setminus \{y'\}$. By Proposition 6.16(3), if $A \cong \mathbf{S}(a)_+$ is an open annulus connected component of $Y \setminus \{y\}$ and $A' \subset Y' \setminus \{y'\}$ is a connected component mapping to A , then we can choose an isomorphism $A' \cong \mathbf{S}(a')_+$ such that $\mathbf{S}(a')_+ \cong A' \rightarrow A \cong \mathbf{S}(a)_+$ is of the form $t \mapsto t^{\delta}$; this map extends to a morphism $\mathbf{B}(1)_+ \rightarrow \mathbf{B}(1)_+$, and these maps glue to give a tame covering $X' \rightarrow X$. \blacksquare

6.18. A local lifting theorem. Let (Y, y, D) be a punctured star-shaped curve with skeleton $\Sigma_0 = \Sigma(Y, \{y\} \cup D)$ and residue curve C_y . Let C' be a smooth, proper, connected k -curve and let $\bar{\varphi} : C' \rightarrow C_y$ be a finite, tamely ramified morphism branched only over the points of C_y corresponding to tangent directions at y represented by edges in Σ_0 . A lifting of C' to a punctured star-shaped curve over (Y, y, D) is the data of a punctured star-shaped curve (Y', y', D') , a tame covering $\varphi : (Y', y', D') \rightarrow (Y, y, D)$, and an isomorphism of the residue curve $C_{y'}$ with C' which identifies $\bar{\varphi}$ with the morphism $\varphi_{y'} : C_{y'} \rightarrow C_y$ induced by φ . An isomorphism between two liftings (Y', y') and (Y'', y'') is a Y -isomorphism $Y' \rightarrow Y''$ such that the induced morphism $C_{y'} \xrightarrow{\sim} C_{y''}$ respects the identifications $C_{y'} \cong C'$ and $C_{y''} \cong C'$.

Theorem 6.19. Let (Y, y, D) be a punctured star-shaped curve with skeleton $\Sigma_0 = \Sigma(Y, \{y\} \cup D)$ and residue curve C_y , let C' be a smooth, proper, connected k -curve, and let $\bar{\varphi} : C' \rightarrow C_y$ be a finite, tamely ramified morphism branched only over the points of C_y corresponding to tangent directions at

y represented by edges in Σ_0 . Then there exists a lifting of C' to a punctured star-shaped curve over (Y, y, D) , and this lifting is unique up to unique isomorphism.

6.20. Before giving the proof of this theorem, we need some technical preliminaries.

Let \mathcal{X} be a finitely presented, flat, separated R -scheme and let \mathfrak{X} be its ϖ -adic completion; this is an admissible formal R -scheme by [BPR11, Proposition 3.12]. There is a canonical open immersion $i_{\mathcal{X}} : \mathfrak{X}_K \rightarrow \mathcal{X}_K^{\text{an}}$ defined in [Con99, §A.3] which is functorial in \mathcal{X} and respects the formation of fiber products, and is an isomorphism when \mathcal{X} is proper over R . (This fact is implicitly contained in the statement of Lemma 5.1.)

Lemma 6.21. *Let $\mathcal{X}, \mathcal{X}'$ be finitely presented, flat, separated R -schemes and let $\mathfrak{X}, \mathfrak{X}'$ denote their ϖ -adic completions. Let $\mathcal{X}' \rightarrow \mathcal{X}$ be a finite and flat morphism. Then the square*

$$(6.21.1) \quad \begin{array}{ccc} \mathfrak{X}'_K & \xrightarrow{i_{\mathcal{X}'}} & \mathcal{X}'_K^{\text{an}} \\ f \downarrow & & \downarrow g \\ \mathfrak{X}_K & \xrightarrow{i_{\mathcal{X}}} & \mathcal{X}_K^{\text{an}} \end{array}$$

is Cartesian.

Proof. The vertical arrows f, g of (6.21.1) are finite and the horizontal arrows $i_{\mathcal{X}'}, i_{\mathcal{X}}$ are open immersions. Let $Y = \mathcal{X}'_K^{\text{an}} \times_{\mathcal{X}_K^{\text{an}}} \mathfrak{X}_K$, so $Y \rightarrow \mathcal{X}'_K^{\text{an}}$ is an open immersion and $Y \rightarrow \mathfrak{X}_K$ is finite. Let $h : \mathfrak{X}'_K \rightarrow Y$ be the canonical morphism. Then h is an open immersion because its composition with $Y \rightarrow \mathcal{X}'_K^{\text{an}}$ is an open immersion, and h is finite because its composition with $Y \rightarrow \mathfrak{X}_K$ is finite. Therefore h is an isomorphism of \mathfrak{X}'_K onto an open and closed subspace of Y . It suffices to show that h is surjective, and since $Y(K)$ is dense in Y , we only need to check that $Y(K)$ is in the image of h .

Let $x \in \mathfrak{X}_K(K)$ and $y \in \mathcal{X}'_K^{\text{an}}(K)$ be points with the same image $z \in \mathcal{X}_K^{\text{an}}(K)$, so $(x, y) \in Y(K)$. Choose an open formal affine $\mathfrak{U} \subset \mathfrak{X}$ such that $x \in \mathfrak{U}_K(K)$, and assume without loss of generality that there is an affine open $\mathcal{U} = \text{Spec}(A) \subset \mathcal{X}$ such that $\mathfrak{U} = \text{Spf}(\hat{A})$, where \hat{A} is the ϖ -adic completion of A . Then x corresponds to a continuous K -homomorphism $\eta : \hat{A} \otimes_R K \rightarrow K$. We have $\eta(\hat{A}) \subset R$, so composing with the completion homomorphism $A \rightarrow \hat{A}$ we obtain an R -homomorphism $A \rightarrow R$. Let $\xi : \text{Spec}(R) \rightarrow \mathcal{U} \subset \mathcal{X}$ denote the induced morphism. The corresponding morphism $\text{Spf}(R)_K \rightarrow \mathfrak{X}_K$ is the point x .

Let \mathcal{Z} be the fiber product of $\mathcal{X}' \rightarrow \mathcal{X}$ with $\eta : \text{Spec}(R) \rightarrow \mathcal{X}$ and let \mathfrak{Z} be its ϖ -adic completion. Note that \mathcal{Z} is finite and flat over $\text{Spec}(R)$. In particular, \mathcal{Z} is proper over R , so $i_{\mathcal{Z}} : \mathfrak{Z}_K \rightarrow \mathcal{Z}_K^{\text{an}}$ is an isomorphism. We have a commutative cube

$$\begin{array}{ccccc} & & \mathcal{Z}_K^{\text{an}} & \xrightarrow{\quad} & \mathcal{X}'_K^{\text{an}} \\ & \nearrow \cong & \downarrow & & \downarrow g \\ \mathfrak{Z}_K & \xrightarrow{\quad} & \mathfrak{X}'_K & & \\ \downarrow & & \downarrow f & & \\ & \nearrow \cong & \text{Spec}(K)^{\text{an}} & \xrightarrow{z} & \mathcal{X}_K^{\text{an}} \\ \text{Spf}(R)_K & \xrightarrow{x} & \mathfrak{X}_K & & \end{array}$$

where the front and back faces are Cartesian and the diagonal arrows are the canonical open immersions. Since $g(y) = z$ and $\mathcal{Z}_K^{\text{an}} = f^{-1}(z)$, the point y lifts to a point in $\mathcal{Z}_K^{\text{an}}(K)$, which then lifts to a point in $\mathfrak{Z}_K(K)$ whose image in $\mathfrak{X}'_K(K)$ maps to x in $\mathfrak{X}_K(K)$ and $y \in \mathcal{X}'_K^{\text{an}}(K)$. ■

6.21.2. Proof of Theorem 6.19. We first prove the theorem when Y is proper. In this case we may and do assume that Y is the analytic generic fiber of a smooth, proper, connected formal curve \mathfrak{X} over $\text{Spf}(R)$. Let $\mathcal{X} \rightarrow \text{Spec}(R)$ be the algebraization of \mathfrak{X} (Lemma 5.1); this is a smooth, proper relative curve of finite presentation and with connected fibers whose ϖ -adic completion is isomorphic to \mathfrak{X} . Note that $\mathcal{X}_k = \mathfrak{X}_k = C_y$. Let $X = \mathcal{X}_K$, so $X^{\text{an}} = \mathfrak{X}_K = Y$. By the valuative criterion of properness,

every point x of D extends uniquely to a section $\mathrm{Spec}(R) \rightarrow \mathcal{X}$ which sends the closed point to the reduction of x ; hence the closure \mathcal{D} of D in \mathcal{X} is a disjoint union of sections. Let $\mathcal{U} = \mathcal{X} \setminus \mathcal{D}$.

The theory of the *tamely ramified étale fundamental group* π_1^t of a morphism of schemes with a relative normal crossings divisor is developed in [SGA1, Exposé XIII]. The finite-index subgroups of π_1^t classify so-called tamely ramified étale covers. The subscheme $\mathcal{D} \subset \mathcal{X}$ is a relative normal crossings divisor relative to $\mathrm{Spec}(R)$, so the proof of Corollaire 2.12 of *loc. cit.* shows that the specialization homomorphism $\pi_1(\mathcal{U}_K) \rightarrow \pi_1^t(\mathcal{U}_k)$ is surjective (we suppress the base points). What this means concretely is that every tamely ramified cover $\overline{U}' \rightarrow \mathcal{U}_k$ over \mathcal{X}_k relative to \mathcal{D}_k extends to a finite étale morphism $U' \rightarrow \mathcal{U}$, unique up to unique isomorphism, and the generic fiber of U' is connected.

Let $\overline{D}' = \overline{\varphi}^{-1}(\mathcal{D}_k)$ and let $\overline{U}' = C' \setminus \overline{D}'$, so $\overline{U}' \rightarrow \mathcal{U}_k$ is a tamely ramified cover of \mathcal{U}_k over \mathcal{X}_k relative to \mathcal{D}_k . Let $\varphi : U' \rightarrow \mathcal{U}$ be the unique étale covering extending $\overline{\varphi}|_{\overline{U}'}$; this is equipped with an isomorphism $\mathcal{U}'_k \cong \overline{U}'$ identifying $\overline{\varphi}|_{\overline{U}'}$ with φ_k . Let $U = \mathcal{U}_K = X \setminus D$, let $U' = \mathcal{U}'_K$, let X' be the smooth compactification of U' , and let $\varphi_K : X' \rightarrow X$ denote the unique morphism extending $\varphi_K : U' \rightarrow U$. We will show that there is a simple neighborhood W of y in X^{an} such that $\varphi_K^{-1}(W) \rightarrow W$ is a tame covering, and conclude that a lifting exists using Corollary 6.17.

6.21.3. First we claim that $\varphi_K^{-1}(y)$ consists of a unique point $y' \in X'^{\mathrm{an}}$. Letting \mathfrak{U} and \mathfrak{U}' denote the ϖ -adic completions of \mathcal{U} and \mathcal{U}' , respectively, we have a Cartesian square

$$\begin{array}{ccc} \mathfrak{U}'_K & \xrightarrow{i_{U'}} & U'^{\mathrm{an}} \\ \downarrow & & \downarrow \\ \mathfrak{U}_K & \xrightarrow{i_U} & U^{\mathrm{an}} \end{array}$$

by Lemma 6.21. In other words, \mathfrak{U}'_K is the inverse image of \mathfrak{U}_K under $\varphi_K : U'^{\mathrm{an}} \rightarrow U^{\mathrm{an}}$. Since the reduction map $\mathfrak{U}_K \rightarrow \mathfrak{U}_k = \mathcal{U}_k$ is surjective and y is the unique point of X^{an} reducing to the generic point of $\mathfrak{U}_k \subset \mathfrak{X}_k$, we have $y \in \mathfrak{U}_K$, so it suffices to show that there is a unique point $y' \in \mathfrak{U}'_K$ mapping to y . It follows easily from the functoriality of the reduction map that the only point y' mapping to y is the unique point of \mathfrak{U}'_K reducing to the generic point of \mathfrak{U}'_k . In particular, the residue curve of X'^{an} at y' is identified with C' , which is the smooth completion of $\mathfrak{U}'_k = \overline{U}'$.

6.21.4. Choose a strongly semistable vertex set V' of X' containing y' , let $\Sigma' = \Sigma(X', V')$, and let $\tau : X'^{\mathrm{an}} \rightarrow \Sigma'$ be the retraction. Let $B \subset X^{\mathrm{an}}$ be the formal fiber of a point $\overline{x} \in \mathfrak{U}_k(k)$; equivalently, B is a connected component of $X^{\mathrm{an}} \setminus \{y\}$ not meeting D . Let B' be a connected component of $X'^{\mathrm{an}} \setminus \{y'\}$ contained in $\varphi_K^{-1}(B)$. We have $B \subset \mathfrak{U}_K$; therefore $B' \subset \mathfrak{U}'_K$, so B' is a connected component of $\mathfrak{U}'_K \setminus \{y'\}$, hence B' is the formal fiber of a point $\overline{x}' \in \mathfrak{U}'_k(k)$, so $B' \cong \mathbf{B}(1)_+$. It follows that $V' \setminus (V' \cap B')$ is again a semistable vertex set, so we may and do assume that $\Sigma' \cap B' = \emptyset$ for all such B' .

Let $e' \subset \Sigma'$ be an edge adjacent to y' and let B be the connected component of $X^{\mathrm{an}} \setminus \{y\}$ containing $\varphi_K(e'^{\circ})$. By the above, B is the formal fiber of a point $\overline{x} \in \mathcal{D}_k$. Fix an isomorphism $B \cong \mathbf{B}(1)_+$ taking the unique point in $D \cap B$ to 0. Let $v' \in T_{y'}$ be the tangent vector in the direction of e' and let $v = d\varphi_K(y')(v') \in T_y$. The point in $\mathfrak{X}_k(k)$ corresponding to v is \overline{x} ; let $\overline{x}' \in \overline{D}' \subset C'(k)$ be the point corresponding to v' , so $\overline{\varphi}(\overline{x}') = \overline{x}$. By Theorem 4.23, the ramification degree δ of $\overline{\varphi}$ at \overline{x}' is equal to $d_{v'}\varphi_K(x')$.

Let B' be the connected component of $X'^{\mathrm{an}} \setminus \{y'\}$ containing e'° . Viewing $\varphi_K|_{B'}$ as a morphism $B' \rightarrow \mathbf{B}(1)_+ \subset \mathbf{A}_K^{1,\mathrm{an}}$ via our chosen isomorphism $B \cong \mathbf{B}(1)_+$, the map $x' \mapsto \log|\varphi_K(x')|$ is a piecewise affine function on e'° which changes slope on the retractions of the zeros of φ_K . Shrinking e' if necessary, we may and do assume that $\log|\varphi_K|$ is affine on e'° ; its slope has absolute value δ , and $\log|\varphi_K(x')| \rightarrow 0$ as $x' \rightarrow y'$. By (4.24) the restriction of φ_K to the open annulus $\tau^{-1}(e'^{\circ})$ is a finite morphism of degree δ onto an open annulus $\mathbf{S}(a)_+ \subset \mathbf{B}(1)_+$, where $\mathrm{val}(a)$ is δ times the length of e'° .

6.21.5. Enlarging Σ' if necessary, we may and do assume that every tangent direction $v' \in T_{y'}$ corresponding to a point in \overline{D}' is represented by an edge in Σ' . Let e'_1, \dots, e'_r be the edges of Σ' adjacent

to y' , for $i = 1, \dots, r$ let B_i be the connected component of $X^{\text{an}} \setminus \{y\}$ containing $\varphi_K(e'_i)$, and choose isomorphisms $B_i \xrightarrow{\sim} \mathbf{B}(1)_+$ sending the unique point of $D \cap B_i$ to 0. (The balls B_1, \dots, B_r are not necessarily distinct; we mean that one should choose a single isomorphism for each distinct ball.) Let $\bar{x}'_i \in \bar{D}'$ be the point corresponding to the tangent direction at y' in the direction of e'_i , and let δ_i be the ramification degree of $\bar{\varphi}$ at \bar{x}'_i . Applying the procedure of (6.21.4) for each e'_i , and shrinking if necessary, we may and do assume that for every i , φ_K induces a degree- δ_i morphism of $\tau^{-1}(e'^{\circ}_i)$ onto an open annulus $\mathbf{S}(a)_+ \subset \mathbf{B}(1)_+ \cong B_i$, with a independent of i .

Let $W \subset X^{\text{an}}$ be the union of $\tau^{-1}(y)$ with the annuli $\mathbf{S}(a)_+ \subset B_i$, so W is obtained from X^{an} by removing a closed ball around each point of D . This is a simple neighborhood of y in X^{an} , hence is a star-shaped curve. Let $\Sigma'_0 = \{y'\} \cup \bigcup_{i=1}^r e'^{\circ}_i$ and let $W' = \tau^{-1}(\Sigma'_0)$, so W' is a simple neighborhood of y' in X'^{an} and is hence a star-shaped curve. We claim that $W' = \varphi_K^{-1}(W)$. Clearly $\varphi_K(W') \subset W$, so it suffices to show that the fibers of φ_K have length equal to $\deg(\varphi_K)$. This is certainly the case for $\varphi_K^{-1}(y) = \{y'\}$. If $B \subset X^{\text{an}}$ is the formal fiber of a point in $\mathfrak{U}_k(k)$, then as in the proof of Proposition 6.16(1), $\varphi_K^{-1}(B) \subset W'$ is a disjoint union of $\deg(\varphi_K)$ open balls mapping isomorphically onto B . For each i the inverse image of $\mathbf{S}(a)_+ \subset \mathbf{B}(1)_+ \cong B_i$ in W' is a disjoint union of annuli, one for each point in the fiber of $\bar{\varphi}$ containing \bar{x}'_i , and the degree of φ_K restricted to each annulus is the ramification degree of $\bar{\varphi}$ at the corresponding point. The sum of the ramification degrees of $\bar{\varphi}$ at the points in any fiber is equal to $\deg(\bar{\varphi}) = \deg(\varphi)$, which proves the claim. Therefore φ_K induces a finite morphism $W' \rightarrow W$, which is a tame covering of (W, y) (really of (W, y, \emptyset)).

By construction, (Y, y) is a compactification of (W, y) , so by Corollary 6.17, the tame covering $W' \rightarrow W$ lifts to a tame covering $Y' \rightarrow Y$ relative to D . It is clear that $(Y', y', \varphi_K^{-1}(D))$ is a lifting of C' to a punctured star-shaped curve over (Y, y, D) . (One can show that in fact $Y' \cong X'^{\text{an}}$, although this is not clear *a priori*.)

6.21.6. It remains to prove (still in the case when Y is proper) that liftings are unique up to unique isomorphism. Let $\varphi_K : (Y', y', D') \rightarrow (Y, y, D)$ and $\varphi'_K : (Y'', y'', D'') \rightarrow (Y, y, D)$ be two liftings of C' to punctured star-shaped curves over (Y, y, D) . Then Y' and Y'' are also proper, hence we may and do assume that they are the analytic generic fibers of smooth, proper, connected formal curves \mathfrak{X}' and \mathfrak{X}'' as in (6.21.2). Since $\varphi_K(y') = \varphi'_K(y'') = y$, there are unique finite morphisms $\varphi : \mathfrak{X}' \rightarrow \mathfrak{X}$ and $\varphi' : \mathfrak{X}'' \rightarrow \mathfrak{X}$ extending φ and φ' , respectively, and φ_K and φ'_K are identified with $\bar{\varphi}$ under the isomorphisms $C_{y'} = \mathfrak{X}'_k \cong C'$ and $C_{y''} = \mathfrak{X}''_k \cong C'$. Let \mathcal{X}' and \mathcal{X}'' be the algebraizations of \mathfrak{X}' and \mathfrak{X}'' , respectively, and let $\mathcal{U}' = \mathcal{X}' \setminus \varphi^{-1}(D)$ and $\mathcal{U}'' = \mathcal{X}'' \setminus \varphi'^{-1}(D)$. Then $\mathcal{U}', \mathcal{U}''$ are finite étale coverings of \mathcal{U} lifting $\bar{\mathcal{U}}' \rightarrow \mathcal{U}_k$, so there is a unique isomorphism $\psi : \mathcal{U}' \xrightarrow{\sim} \mathcal{U}''$ over \mathcal{U} . Since Y' (resp. Y'') is the analytification of the smooth completion of \mathcal{U}'_K (resp. \mathcal{U}''_K), ψ_K extends uniquely to a Y -isomorphism $\psi_K : Y' \xrightarrow{\sim} Y''$; we have $\psi_K(y') = y''$, and $\psi_{y'} : C_{y'} \xrightarrow{\sim} C_{y''}$ is a C_y -isomorphism because $\mathcal{U}'_k \xrightarrow{\sim} \mathcal{U}''_k$ is a \mathcal{U}_k -isomorphism. Hence ψ_K is an isomorphism of liftings.

As for uniqueness, suppose that $\psi_K : (Y', y') \rightarrow (Y', y')$ is an automorphism of Y' as a lifting of C' . Then ψ_K extends uniquely to an \mathcal{X} -automorphism $\psi : \mathcal{X}' \xrightarrow{\sim} \mathcal{X}'$ that is the identity on \mathcal{X}'_k , which restricts to a \mathcal{U} -automorphism of \mathcal{U}' that is the identity on \mathcal{U}'_k . It follows from the uniqueness of \mathcal{U}' up to unique isomorphism that ψ_K is the identity when restricted to \mathcal{U}'_K , so since Y' is the analytification of the smooth completion of \mathcal{U}'_K , the automorphism ψ_K is the identity. This concludes the proof for proper Y .

6.21.7. Now suppose that (Y, y) is not proper. Let $Y \hookrightarrow X$ be a compactification (6.6) of Y ; we will identify Y with its image in X . Let D_1 be the union of D with a choice of one K -point from every connected component of $X \setminus Y$. Let $\varphi : (X', y', D'_1) \rightarrow (X, y, D_1)$ be a lifting of C' to a punctured star-shaped curve over (X, y, D_1) and let $Y' = \varphi^{-1}(Y)$. Then $(Y', y', \varphi^{-1}(D))$ is a lifting of C' to a punctured star-shaped curve over (Y, y, D) .

Let $\psi : Y' \rightarrow Y'$ be an automorphism of Y' as a lifting of C' . Since ψ induces the identity map $C_{y'} \rightarrow C_{y'}$ on residue curves, ψ takes each connected component of $Y' \setminus \{y'\}$ to itself. Recall that $X' \setminus Y'$ consists of a disjoint union of closed balls around the points of $D'_1 \setminus D'$, where $D'_1 = \varphi^{-1}(D_1)$. Let $x' \in D'_1 \setminus D'$, let B' be the connected component of $X' \setminus \{y'\}$ containing x' , and let $B = \varphi(B') \subset X$.

By Proposition 6.16(2), we can choose isomorphisms $B \cong \mathbf{B}(1)_+$ and $B' \cong \mathbf{B}(1)_+$ sending x' and $\varphi(x')$ to 0, such that the composition

$$\mathbf{B}(1)_+ \cong B' \xrightarrow{\varphi} B \cong \mathbf{B}(1)_+$$

is of the form $t \mapsto t^\delta$. Let $A' = Y' \cap B'$ and let $A = Y \cap B = \varphi(A')$. The isomorphism $B \cong \mathbf{B}(1)_+$ (resp. $B' \cong \mathbf{B}(1)_+$) identifies A (resp. A') with an open annulus $\mathbf{S}(a)_+$ (resp. $\mathbf{S}(a')_+$) in $\mathbf{B}(1)_+$. Since $\psi|_{A'} : A' \rightarrow A$ is an A -morphism, the composition $\mathbf{S}(a')_+ \cong A' \rightarrow A \cong \mathbf{S}(a)_+$ is of the form $t \mapsto \zeta t$, where $\zeta \in R^\times$ is a δ th root of unity. Therefore $\psi|_{A'}$ extends uniquely to a B -morphism $\psi|_{B'} : B' \rightarrow B$ fixing 0. Gluing these morphisms together, we obtain an X -morphism $\psi : X' \rightarrow X$ extending $\psi : Y' \rightarrow Y$. By construction this is an automorphism of X' as a lifting of C' to X , which is thus the identity. Therefore $\psi : Y' \rightarrow Y$ is the identity.

Let $\varphi' : (Y'', y'', D'') \rightarrow (Y, y, D)$ be another lifting of C' to a punctured star-shaped curve over (Y, y, D) . By Corollary 6.17, there exists a compactification $Y'' \hookrightarrow X''$ of Y'' and an extension of φ' to a tame covering $\varphi' : (X'', y'', \varphi'^{-1}(D_1)) \rightarrow (X, y, D_1)$. This is another lifting of C' to a punctured star-shaped curve over (X, y, D_1) , so there is an isomorphism $\psi : (X', y') \xrightarrow{\sim} (X'', y'')$ of liftings. Since $Y' = \varphi^{-1}(Y)$ and $Y'' = \varphi'^{-1}(Y)$, the isomorphism ψ restricts to an isomorphism $(Y', y') \xrightarrow{\sim} (Y'', y'')$ of liftings. ■

Corollary 6.22. *With the notation in Theorem 6.19, let $(Y', y', D') \rightarrow (Y, y, D)$ be a lifting of C' to a punctured star-shaped curve over (Y, y, D) . Then the natural homomorphism*

$$\mathrm{Aut}_Y(Y') \longrightarrow \mathrm{Aut}_{C_y}(C')$$

is bijective. If $(Y'', y'', D'') \rightarrow (Y, y, D)$ is a second lifting of C' to a punctured star-shaped curve over (Y, y, D) then the natural map

$$\mathrm{Isom}_Y(Y'', Y') \longrightarrow \mathrm{Isom}_{C_y}(C_{y''}, C_{y'})$$

is bijective.

7. CLASSIFICATION OF LIFTINGS OF HARMONIC MORPHISMS OF METRIZED COMPLEXES

Fix a triangulated punctured curve $(X, V \cup D)$ with skeleton $\Sigma = \Sigma(X, V \cup D)$ and let $\tau : X^{\mathrm{an}} \rightarrow \Sigma$ be the canonical retraction. Throughout this section, we **assume that Σ has no loop edges**. Let $\varphi : \Sigma' \rightarrow \Sigma$ be a tame covering of metrized complexes of curves. A *lifting of Σ' to a tame covering of $(X, V \cup D)$* is a tame covering of triangulated punctured curves $\varphi : (X', V' \cup D') \rightarrow (X, V \cup D)$ (see (4.30)) equipped with a Σ -isomorphism $\varphi^{-1}(\Sigma) \cong \Sigma'$ of metrized complexes of curves. Since $V' = \varphi^{-1}(V)$ and $D' = \varphi^{-1}(D)$, we will often denote a lifting simply by X' . Let $\varphi : X' \rightarrow X$ and $\varphi' : X'' \rightarrow X$ be two liftings of Σ' and let $\psi : X' \xrightarrow{\sim} X''$ be an X -isomorphism of curves. Then ψ restricts to a Σ -automorphism

$$\psi|_{\Sigma'} : \Sigma' \cong \varphi^{-1}(\Sigma) \xrightarrow{\sim} \varphi'^{-1}(\Sigma) \cong \Sigma'.$$

We will consider liftings up to X -isomorphism preserving Σ' , i.e. such that $\psi_{\Sigma'}$ is the identity, and we will also consider liftings up to isomorphism as curves over X .

7.1. For every finite vertex $x \in V$, let $\Sigma(x)$ be the connected component of x in $\{x\} \cup (\Sigma \setminus V)$, let $Y(x) = \tau^{-1}(\Sigma(x))$, and let $D(x) = D \cap Y(x)$, as in Example 6.15. Then $(Y(x), x, D(x))$ is a punctured star-shaped curve. By Proposition 2.25, for every finite vertex $x' \in V(\Sigma')$ lying above x the morphism $\varphi_{x'} : C_{x'} \rightarrow C_x$ is finite, tamely ramified, and branched only over the points of C_x corresponding to tangent directions at x represented by edges of $\Sigma(x)$. Let $\psi(x') : (Y'(x'), x', D(x')) \rightarrow (Y(x), x, D(x))$ be the unique lifting of $C_{x'}$ to a punctured star-shaped curve over $(Y(x), x, D(x))$ provided by Theorem 6.19 and let $\Sigma'(x') = \psi(x')^{-1}(\Sigma(x)) = \Sigma(Y', \{x'\} \cup D'(x'))$. Then $\Sigma'(x')$ is canonically identified with the connected component of x' in $\{x'\} \cup (\Sigma' \setminus V_f(\Sigma'))$ in such a way that for every edge e' of $\Sigma'(x')$, the point $\mathrm{red}_{x'}(e') \in C_{x'}(k)$ is identified with the point $\mathrm{red}(e')$ defined in (6.7). This induces an identification of $D'(x')$ with $\varphi^{-1}(D) \cap \Sigma'(x')$. Let $\tau_{x'}$ be the canonical retraction $Y'(x') \rightarrow \Sigma'(x')$ defined in (6.3).

7.2. Let $e' \in E_f(\Sigma')$ and $e = \varphi(e')$. Choose $a, a' \in K^\times$ with $\text{val}(a) = \ell(e)$ and $\text{val}(a') = \ell(e')$. Let x', y' be the endpoints of e' . By Proposition 6.16(3), we can choose isomorphisms $\tau_{x'}^{-1}(e'^\circ) \cong \mathbf{S}(a)_+$ and $\tau_{y'}^{-1}(e'^\circ) \cong \mathbf{S}(a')_+$, $\tau_{x'}^{-1}(e'^\circ) \cong \mathbf{S}(a')_+$ in such a way that the finite morphisms $\tau_{x'}^{-1}(e'^\circ) \rightarrow \tau^{-1}(e^\circ)$ and $\tau_{y'}^{-1}(e'^\circ) \rightarrow \tau^{-1}(e^\circ)$ are given by $t \mapsto t^{d_{e'}(\varphi)}$, where $d_{e'}(\varphi)$ is the degree of the edge map $e' \rightarrow e$. In particular, there exists a $\tau^{-1}(e^\circ)$ -isomorphism $\tau_{x'}^{-1}(e'^\circ) \xrightarrow{\sim} \tau_{y'}^{-1}(e'^\circ)$. As explained in (6.11), there are canonical identifications

$$\text{Aut}_{\tau^{-1}(e^\circ)}(d_{e'}(\varphi)) := \text{Aut}_{\tau^{-1}(e^\circ)}(\tau_{x'}^{-1}(e'^\circ)) = \text{Aut}_{\tau^{-1}(e^\circ)}(\tau_{y'}^{-1}(e'^\circ)) \cong \mathbf{Z}/d_{e'}(\varphi)\mathbf{Z};$$

hence the set of isomorphisms $\tau_{x'}^{-1}(e'^\circ) \xrightarrow{\sim} \tau_{y'}^{-1}(e'^\circ)$ is a principal homogeneous space under the pre- or post-composition action of $\text{Aut}_{\tau^{-1}(e^\circ)}(d_{e'}(\varphi))$.

Let $E_f^\pm(\Sigma')$ denote the set of oriented finite edges of Σ' , and for $e' \in E_f^\pm(\Sigma')$ let \bar{e}' denote the same edge with the opposite orientation. Let $\mathcal{G}(\Sigma', X)$ denote the set of tuples $(\Theta_{e'})_{e' \in E_f^\pm(\Sigma')}$ of isomorphisms

$$(7.2.1) \quad \Theta_{e'} : \tau_{x'}^{-1}(e'^\circ) \xrightarrow{\sim} \tau_{y'}^{-1}(e'^\circ)$$

such that $\Theta_{\bar{e}'} = \Theta_{e'}^{-1}$, where $e' = \overrightarrow{x'y'}$. We call $\mathcal{G}(\Sigma', X)$ the set of *gluing data* for a lifting of Σ' to a tame covering of $(X, V \cup D)$, and we emphasize that $\mathcal{G}(\Sigma', X)$ is *nonempty*.

7.3. Let $\alpha \in \text{Aut}_\Sigma(\Sigma')$, so α is a degree-1 finite harmonic morphism $\Sigma' \xrightarrow{\sim} \Sigma'$ preserving $\Sigma' \rightarrow \Sigma$. Let $x' \in V_f(\Sigma')$ and let $x'' = \alpha(x')$ and $x = \varphi(x') = \varphi(x'')$. Part of the data of α is a C_x -isomorphism $\alpha_{x'} : C_{x'} \xrightarrow{\sim} C_{x''}$. By Corollary 6.22 there is a unique lift of $\alpha_{x'}$ to a $Y(x)$ -isomorphism $Y'(x') \xrightarrow{\sim} Y'(x'')$ of punctured star-shaped curves inducing the isomorphism $C_{x'} \xrightarrow{\sim} C_{x''}$ on residue curves. Let $e' \in E_f(\Sigma')$ be an edge adjacent to x' , let $e'' = \alpha(e')$, and let $e = \varphi(e') = \varphi(e'')$. Then $\alpha_{x'}$ restricts to an isomorphism

$$\alpha_{x'} : \tau_{x'}^{-1}(e'^\circ) \xrightarrow{\sim} \tau_{x''}^{-1}(e''^\circ).$$

Define the *conjugation action* of $\text{Aut}_\Sigma(\Sigma')$ on $\mathcal{G}(\Sigma', X)$ by the rule

$$(7.3.1) \quad \alpha \cdot (\Theta_{e'})_{e' \in E_f^\pm(\Sigma')} = (\alpha_{y'}^{-1} \circ \Theta_{\alpha(e')} \circ \alpha_{x'})_{e' \in E_f^\pm(\Sigma')}$$

where $e' = \overrightarrow{x'y'}$.

Theorem 7.4. (Classification of lifts of harmonic morphisms) *Let $(X, V \cup D)$ be a triangulated punctured curve with skeleton $\Sigma = \Sigma(X, V \cup D)$. Assume that Σ has no loop edges. Let $\varphi : \Sigma' \rightarrow \Sigma$ be a tame covering of metrized complexes of curves.*

- (1) *There is a canonical bijection between the set of gluing data $\mathcal{G}(\Sigma', X)$ and the set of liftings of Σ' to a tame covering of $(X, V \cup D)$, up to X -isomorphism preserving Σ' . In particular, there exists a lifting of Σ' . Any such lifting has no nontrivial automorphisms which preserve Σ' .*
- (2) *Two tuples of gluing data determine X -isomorphic curves if and only if they are in the same orbit under the conjugation action (7.3.1). The stabilizer in $\text{Aut}_\Sigma(\Sigma')$ of an element of $\mathcal{G}(\Sigma', X)$ is canonically isomorphic to the X -automorphism group of the associated curve.*

Proof. Given $(\Theta_{e'}) \in \mathcal{G}(\Sigma', X)$ one can glue the local lifts $\{Y(x')\}_{x' \in V_f(\Sigma')}$ via the isomorphisms (7.2.1) to obtain an analytic space which one easily verifies is smooth and proper, hence arises as the analytification of an algebraic curve X' . Moreover the morphisms $Y'(x') \rightarrow Y(x)$ glue to give a morphism $X'^{\text{an}} \rightarrow X^{\text{an}}$, which is the analytification of a morphism $\varphi : X' \rightarrow X$. By construction, if $V' = \varphi^{-1}(V)$ and $D' = \varphi^{-1}(D)$, then $(X', V' \cup D') \rightarrow (X, V \cup D)$ is a lifting of Σ' to a tame covering of $(X, V \cup D)$.

Now let $\varphi : (X', V' \cup D') \rightarrow (X, V \cup D)$ be any lifting of Σ' to a tame covering of $(X, V \cup D)$. As explained in Example 6.15, for every $x \in V$ the inverse image $\varphi^{-1}(Y(x))$ is a disjoint union of tame covers of the punctured star-shaped curve $(Y(x), x, D(x))$, one for each $x' \in \varphi^{-1}(x)$. By Theorem 6.19, we have canonical identification $\varphi^{-1}(Y(x)) = \coprod_{x' \mapsto x} Y(x')$. For $x', y' \in V'$ we have

$Y(x') \cap Y(y') = \coprod_{e' \ni x', y'} \tau^{-1}(e'^{\circ})$, where $\tau : X'^{\text{an}} \rightarrow \Sigma'$ is the retraction. Hence X'^{an} is obtained by pasting the local liftings $Y(x')$ via a choice of isomorphisms $(\Theta_{e'}) \in \mathcal{G}(\Sigma', X)$.

Let $\psi : X' \xrightarrow{\sim} X'$ be an X -automorphism preserving Σ' . Since ψ is the identity on the set $\Sigma' \subset X'^{\text{an}}$, for $x' \in V'$ we have $\psi(Y(x')) = Y(x')$. We also have that for $x' \in V'$ the map of residue curves $\psi_{x'} : C_{x'} \rightarrow C_{x'}$ is the identity, so by Corollary 6.22, ψ restricts to the identity morphism $Y(x') \rightarrow Y(x')$. Since $X'^{\text{an}} = \bigcup_{x' \in V'} Y(x')$ we have that ψ is the identity. It follows from this that the tuple $(\Theta_{e'})$ can be recovered from the class of X' modulo X -isomorphisms preserving Σ' , so different gluing data give rise to curves which are not equivalent under such isomorphisms. This proves (1).

Let $\varphi : (X', V' \cup D') \rightarrow (X, V \cup D)$ and $\varphi'' : (X'', V'' \cup D'') \rightarrow (X, V \cup D)$ be the liftings of Σ' associated to the tuples of gluing data $(\Theta'_{e'}), (\Theta''_{e'}) \in \mathcal{G}(\Sigma', X)$, respectively. Suppose that there exists $\alpha \in \text{Aut}_{\Sigma}(\Sigma')$ such that $\alpha \cdot (\Theta'_{e'}) = (\Theta''_{e'})$. Then for all $e' = \overrightarrow{x'y'} \in E_f^{\pm}(\Sigma')$ we have

$$\Theta''_{\alpha(e')} \circ \alpha_{x'} = \alpha_{y'} \circ \Theta'_{e'},$$

so the $Y(\varphi(x'))$ -isomorphisms $\alpha_{x'} : Y(x') \xrightarrow{\sim} Y(\alpha(x'))$ glue to give an X -isomorphism $\alpha : X' \xrightarrow{\sim} X''$. Conversely, the restriction of an X -isomorphism $\alpha : X' \xrightarrow{\sim} X''$ to Σ' is a Σ -automorphism of Σ' . It is easy to see that these are inverse constructions. Taking $X' = X''$ we have an injective homomorphism $\text{Aut}_X(X') \hookrightarrow \text{Aut}_{\Sigma}(\Sigma')$; it follows formally from the above considerations that its image is the stabilizer of $(\Theta'_{e'})$. ■

Remark 7.5. Let X' be a lifting of Σ' to a tame covering of $(X, V \cup D)$. It follows from Theorem 7.4(1) that the natural homomorphism

$$\text{Aut}_X(X') \longrightarrow \text{Aut}_{\Sigma}(\Sigma')$$

is injective. It is not in general surjective: see Example 7.6 (where all the stabilizer groups of elements of $\mathcal{G}(\Sigma', X)$ are proper subgroups of $\text{Aut}_{\Sigma}(\Sigma')$).

We now indicate how Theorem 7.4 implies Theorem 1.4 from the Introduction.

Proof. (Proof of Theorem 1.4) The only issue is that in Theorem 1.4, we merely require that V is a semistable vertex set whereas in Theorem 7.4 we require it to be strongly semistable. However, we can modify Σ by inserting a (valence 2) vertex v along each loop edge, with $C_v \cong \mathbf{P}_k^1$, and with the marked points on C_v being 0 and ∞ . Call the resulting metrized complex $\tilde{\Sigma}$, and let \bar{V}_0 denote the set of vertices which have been added to the vertex set \bar{V} for Σ . We construct a new metrized complex $\tilde{\Sigma}'$ from Σ' by adding $\bar{V}'_0 := \bar{\varphi}^{-1}(\bar{V}_0)$ to the vertex set \bar{V}' for Σ' and letting $C'_{v'} \cong \mathbf{P}_k^1$ for $v' \in \bar{V}'_0$, with the marked points on $C'_{v'}$ being 0 and ∞ . We can extend the tame covering $\bar{\varphi} : \Sigma' \rightarrow \Sigma$ to a tame covering $\tilde{\Sigma}' \rightarrow \tilde{\Sigma}$ by letting the map from $C'_{v'}$ to $C_{\bar{\varphi}(v')}$, for $v' \in \bar{V}'_0$, be $z \mapsto z^d$, where d is the local degree of $\bar{\varphi}$ along the loop edge corresponding to v' . By Theorem 7.4, there is a tame covering $(X', V' \cup \bar{V}'_0 \cup D') \rightarrow (X, V \cup \bar{V}_0 \cup D)$ lifting $\tilde{\Sigma}' \rightarrow \tilde{\Sigma}$, where V'_0 (resp. V_0) corresponds to \bar{V}'_0 (resp. \bar{V}_0). Removing the vertices in V'_0 and V_0 gives a tame covering $(X', V' \cup D') \rightarrow (X, V \cup D)$ lifting $\bar{\varphi}$. ■

Example 7.6. In this example we suppose that $\text{char}(k) \neq 2$. Let E be a Tate curve over K , let Σ be its (set-theoretic) skeleton, and let $\tau : E^{\text{an}} \rightarrow \Sigma$ be the canonical retraction. Let $U : \mathbf{G}_m^{\text{an}}/q^{\mathbf{Z}} \xrightarrow{\sim} E^{\text{an}}$ be the Tate uniformization of E^{an} . The 2-torsion subgroup of E is $U(\{\pm 1, \pm \sqrt{q}\})$; choose a square root of q , let $y = U(1)$ and $z = U(\sqrt{q})$, and let $V = \{y, z\} \subset \Sigma$. This is a semistable vertex set of E and $\Sigma = \Sigma(E, V)$ is the circle with circumference $\text{val}(q)$; the points y and z are antipodal on Σ . Let e_1, e_2 be the edges of Σ ; orient e_1 so that y is the source vertex and e_2 so that z is the source vertex. The residue curves C_y and C_z are both isomorphic to \mathbf{P}_k^1 ; fix isomorphisms $C_y \cong \mathbf{P}_k^1$ and $C_z \cong \mathbf{P}_k^1$ such that the tangent direction at a vertex in the direction of the outgoing (resp. incoming) edge corresponds to ∞ (resp. 0).

Let Σ' be a circle of circumference $\frac{1}{2} \text{val}(q)$, let $V' = \{y', z'\}$ be a pair of antipodal points on Σ' , and let $e'_1 = \overrightarrow{y'z'}$ and $e'_2 = \overrightarrow{z'y'}$ be the two edges of Σ' , with the indicated orientations. Enrich Σ' with the structure of a metrized complex of curves by setting $C_{y'} = C_{z'} = \mathbf{P}_k^1$ and letting the tangent direction

at a vertex in the direction of the outgoing (resp. incoming) edge correspond to ∞ (resp. 0). Define a morphism $\varphi : \Sigma' \rightarrow \Sigma$ of metric graphs by

$$\varphi(y') = y, \quad \varphi(z') = z, \quad \varphi(e'_1) = e_1, \quad \varphi(e'_2) = e_2,$$

with both degrees equal to 2. See Figure 7.

Topologically, $\Sigma' \rightarrow \Sigma$ is a homeomorphism. Let $\varphi_{y'} : C_{y'} \rightarrow C_y$ and $\varphi_{z'} : C_{z'} \rightarrow C_z$ both be the morphism $t \mapsto t^2 : \mathbf{P}_k^1 \rightarrow \mathbf{P}_k^1$. This makes φ into a degree-2 tame covering of metrized complexes of curves.

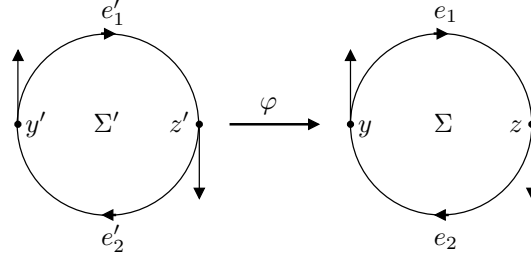


FIGURE 7. This figure illustrates Example 7.6. The morphism φ is a homeomorphism of underlying sets but has a degree of 2 on e'_1 and e'_2 . The arrows on e_1, e_2, e'_1, e'_2 represent the chosen orientations and the indicated tangent vectors at y, z, y', z' represent the tangent direction corresponding to ∞ in the corresponding k -curve.

For $x \in V$ the analytic space $Y(x)$ is an open annulus of logarithmic modulus $\text{val}(q)$; fix an isomorphism $Y(x) \cong \mathbf{S}(q)_+$ for each x . For $x' \in V'$ the analytic space $Y'(x')$ is an open annulus of logarithmic modulus $\frac{1}{2} \text{val}(q)$; fix isomorphisms $Y'(x') \cong \mathbf{S}(\sqrt{q})_+$ such that the morphisms $Y'(x') \rightarrow Y(\varphi(x'))$ are given by $t \mapsto t^2 : \mathbf{S}(\sqrt{q})_+ \rightarrow \mathbf{S}(q)_+$. For $e' \in E(\Sigma')$ we have $d_\varphi(e') = 2$ by definition. Hence $\mathcal{G}(\Sigma', E)$ has four elements, which we label $(\pm 1, \pm 1)$. By Theorem 7.4(1) there are four corresponding classes of liftings of Σ' to a tame covering of (E, V) up to isomorphism preserving Σ' .

Any Σ -automorphism of Σ' is the identity on the underlying topological space. Hence an element of $\text{Aut}_\Sigma(\Sigma')$ is a pair of automorphisms $(\psi_{y'}, \psi_{z'}) \in \text{Aut}_{C_y}(C_{y'}) \times \text{Aut}_{C_z}(C_{z'})$ such that $\psi_{y'}, \psi_{z'}$ fix the points $0, \infty \in \mathbf{P}_k^1$. Thus $\psi_{y'}, \psi_{z'} = \pm 1$, so $\text{Aut}_\Sigma(\Sigma') = \{\pm 1\} \times \{\pm 1\}$. For $x' = y', z'$ the automorphism $-1 : C_{x'} \xrightarrow{\sim} C_{x'}$ lifts to the automorphism $-1 : \mathbf{S}(\sqrt{q})_+ \xrightarrow{\sim} \mathbf{S}(\sqrt{q})_+$. Therefore the conjugation action of $\text{Aut}_\Sigma(\Sigma')$ on $\mathcal{G}(\Sigma', E)$ is given as follows:

$$\begin{aligned} (1, 1) \cdot (\pm 1, \pm 1) &= (-1, -1) \cdot (\pm 1, \pm 1) = (\pm 1, \pm 1) \\ (-1, 1) \cdot (\pm 1, \pm 1) &= (1, -1) \cdot (\pm 1, \pm 1) = -(\pm 1, \pm 1). \end{aligned}$$

By Theorem 7.4(2), there are two isomorphism classes of lifts of Σ' to a tame covering of (E, V) , and each such lift has two automorphisms.

These liftings can be described concretely as follows. Fix a square root of q , let E_\pm be the algebraization of the analytic elliptic curve $\mathbf{G}_m^{\text{an}}/(\pm\sqrt{q})^{\mathbf{Z}}$, and let $\psi_\pm : E_\pm \rightarrow E$ be the morphism $t \mapsto t^2$ on uniformizations. Let $\Sigma_\pm = \psi_\pm^{-1}(\Sigma)$. Then Σ_\pm is isomorphic to Σ' as a tame covering of Σ and E_\pm is a lifting of Σ' to a tame covering of (E, D) . The elliptic curves E_\pm are not isomorphic (as K -schemes) because they have different q -invariants, so they represent the two isomorphism classes of liftings of Σ' . The nontrivial automorphism of E_\pm is given by translating by the image of $-1 \in \mathbf{G}_m^{\text{an}}(K)$ (this is not a homomorphism). In fact, since $\psi_\pm : E_\pm \rightarrow E$ is an étale Galois cover of degree 2, this is the only nontrivial automorphism of E_\pm as an E -curve, so the homomorphism

$$\text{Aut}_E(E_\pm) \longrightarrow \text{Aut}_\Sigma(\Sigma') \cong \{\pm 1\} \times \{\pm 1\}$$

is injective but not surjective (its image is $\{\pm(1, 1)\}$).

7.7. Consider now the automorphism group $\text{Aut}_\Sigma^0(\Sigma') \subset \text{Aut}_\Sigma(\Sigma')$ consisting of all degree-1 finite harmonic morphisms $\alpha : \Sigma' \rightarrow \Sigma'$ respecting $\varphi : \Sigma' \rightarrow \Sigma$ and inducing the identity on the metric graph Γ' underlying Σ' . The restriction of the conjugacy action of $\text{Aut}_\Sigma(\Sigma')$ on $\mathcal{G}(\Sigma', X)$ to the subgroup $\text{Aut}_\Sigma^0(\Sigma')$ admits a simplified description that we describe now. Combining this with arguments similar to those in the proof of Theorem 7.4 yields a classification of the set of liftings of Σ' up to isomorphism as liftings of the metric graph underlying Σ' ; see Theorem 7.8.

First, for each $x' \in V(\Sigma')$ with image $x = \varphi(x')$, let $\text{Aut}_{C_x}^0(C_{x'})$ be the subgroup of $\text{Aut}_{C_x}(C_{x'})$ that fixes every point of $C_{x'}$ of the form $\text{red}(e')$ for some edge e' of Σ' adjacent to x' . Then

$$\text{Aut}_\Sigma^0(\Sigma') = \prod_{x' \in V_f(\Sigma')} \text{Aut}_{C_{\varphi(x)}}^0(C_{x'}) =: \mathcal{E}^0.$$

Denote by \mathcal{E}^1 the finite abelian group

$$\mathcal{E}^1 = \prod_{e' \in E_f(\Sigma')} \text{Aut}_{\tau^{-1}(\varphi(e')^\circ)}(d_{e'}(\varphi)).$$

The discussion preceding Theorem 7.4 shows that the set of gluing data $\mathcal{G}(\Sigma', X)$ is canonically a principal homogeneous space under \mathcal{E}^1 .

The subgroup $\text{Aut}_{C_x}^0(C_{x'}) \subset \text{Aut}_{C_x}(C_{x'})$ corresponds to the subgroup $\text{Aut}_{Y(x)}^0(Y'(x'))$ of automorphisms in $\text{Aut}_{Y(x)}(Y'(x'))$ which act trivially on the skeleton $\Sigma'(x')$. Restriction of a $Y(x)$ -automorphism of $Y'(x')$ to $\tau_{x'}^{-1}(e'^\circ)$ defines a homomorphism

$$\rho_{x', e'} : \text{Aut}_{C_x}^0(C_{x'}) = \text{Aut}_{Y(x)}^0(Y'(x')) \longrightarrow \text{Aut}_{\tau^{-1}(e'^\circ)}(d_{e'}(\varphi)).$$

Fix an orientation of each finite edge of Σ' . For $x' \in V_f(\Sigma')$ with image $x = \varphi(x')$, let $\rho_{x'} : \text{Aut}_{C_x}^0(C_{x'}) \rightarrow \mathcal{E}^1$ be the homomorphism whose e' -coordinate is

$$(\rho_{x'}(\alpha))_{e'} = \begin{cases} \rho_{x', e'}(\alpha) & \text{if } x' \text{ is the source vertex of } e' \\ \rho_{x', e'}(\alpha)^{-1} & \text{if } x' \text{ is the target vertex of } e' \\ 1 & \text{if } x' \text{ is not an endpoint of } e'. \end{cases}$$

Taking the product over all $x' \in V_f(\Sigma')$ yields a homomorphism $\rho : \mathcal{E}^0 \rightarrow \mathcal{E}^1$. The kernel and cokernel of ρ are independent of the choice of orientations. Viewing $\mathcal{E}^0 \rightarrow \mathcal{E}^1$ as a two-term complex \mathcal{E}^\bullet of groups, its cohomology groups are

$$H^0(\mathcal{E}^\bullet) = \ker(\rho) \quad \text{and} \quad H^1(\mathcal{E}^\bullet) = \text{coker}(\rho).$$

Theorem 7.8. *Let $(X, V \cup D)$ be a triangulated punctured curve with skeleton $\Sigma = \Sigma(X, V \cup D)$. Assume that Σ has no loop edges. Let $\varphi : \Sigma' \rightarrow \Sigma$ be a tame covering of metrized complexes of curves.*

- (1) *$\mathcal{G}(\Sigma', X)$ is canonically a principal homogeneous space under \mathcal{E}^1 and the conjugacy action of $\text{Aut}_\Sigma^0(\Sigma')$ on $\mathcal{G}(\Sigma', X)$ is given by the action of the subgroup $\rho(\text{Aut}_\Sigma^0(\Sigma')) = \rho(\mathcal{E}^0) \subseteq \mathcal{E}^1$ on $\mathcal{G}(\Sigma', X)$.*
- (2) *The set of liftings of Σ' up to isomorphism as liftings of the metric graph underlying Σ' is a principal homogeneous space under $H^1(\mathcal{E}^\bullet)$, and the group of automorphisms of a given lifting as a lifting of the metric graph underlying Σ' is isomorphic to $H^0(\mathcal{E}^\bullet)$.*

7.9. Descent to a general ground field. Let K_0 be a subfield of K , let X_0 be a smooth, projective, geometrically connected K_0 -curve, and let $D \subset X_0(K_0)$ be a finite set. Let $X = X_0 \otimes_{K_0} K$, let V be a strongly semistable vertex set of (X, D) , let $\Sigma = \Sigma(X, V \cup D)$, and let $\varphi : \Sigma' \rightarrow \Sigma$ be a tame covering of metrized complexes of curves, as in the statement of Theorem 1.4. Let $\varphi : X' \rightarrow X$ be a lifting of φ to a tame covering of $(X, V \cup D)$. Whereas we take the data of the morphism $\Sigma' \rightarrow \Sigma$ to be geometric, i.e. only defined over K , the covering $X' \rightarrow X$ is in fact defined over a finite, separable extension of K_0 . This follows from the fact that if $U = X \setminus D$ and $U' = X' \setminus \varphi^{-1}(D)$, then $\varphi : U' \rightarrow U$ is a tamely ramified cover of U over X relative to D (see Remarks 6.12(1) and 4.32(1)), along with the following lemma.

Lemma 7.10. *Let K_0 be any field, let K be a separably closed field containing K_0 , let X_0 be a smooth, projective, geometrically connected K_0 -curve, and let $D \subset X_0(K_0)$ be a finite set. Let $X = X_0 \otimes_{K_0} K$, let $\varphi : X' \rightarrow X$ be a finite morphism with X' smooth and (geometrically) connected, and suppose that φ is branched only over D , with all ramification degrees prime to the characteristic of K . Then there exists a finite, separable extension K_1 of K and a morphism $\varphi_1 : X'_1 \rightarrow X_0 \otimes_{K_0} K_1$ descending φ .*

Proof. Let $U_0 = X_0 \setminus D$, and let $U = X \setminus D$ and $U' = X' \setminus \varphi^{-1}(D)$. First suppose that K_0 is separably closed. By [SGA1, Exposé XIII, Corollaire 2.12], the tamely ramified étale fundamental groups $\pi_1^t(U_0)$ and $\pi_1^t(U)$ are isomorphic (with respect to some choice of base point). Since $\varphi : U' \rightarrow U$ is a tamely ramified cover of U over X relative to D , it is classified by a finite-index subgroup of $\pi_1^t(U) = \pi_1^t(U_0)$, so there exists a tamely ramified cover $\varphi_0 : U'_0 \rightarrow U_0$ of U_0 over X_0 relative to D descending φ .

Now we drop the hypothesis that K_0 is separably closed. By the previous paragraph we may assume that K is a separable closure of K_0 . By general principles the projective morphism $X' \rightarrow X$ descends to a subfield of K which is finitely generated (i.e. finite) over K_0 . ■

Remark 7.11. With the notation in (7.9), suppose that K_0 is a complete valued field with value group $\Lambda_0 = \text{val}(K_0^\times)$ and **algebraically closed** residue field k , that Σ is “rational over K_0 ” in that it comes from a (split) semistable formal R_0 -model of X_0 in the sense of Section 5, and that Σ' has edge lengths contained in Λ_0 . With some extra work it is possible to carry out the gluing arguments of Theorem 1.4 directly over the field K_0 (in this context Lemma 6.10(1) still holds), which shows that the cover $X' \rightarrow X$ is in fact defined over K_0 . In the case of a discrete valuation this also follows from [Wew99], or from [Sai97] if $D = \emptyset$.

7.12. Liftings of tame harmonic morphisms. Theorem 1.4 implies the existing of liftings for (finite) tame harmonic morphisms of metrized complexes (which are not necessarily tame coverings, the difference being generic étaleness). See Definition 2.24 for both definitions.

Proposition 7.13. *Let $(X, V \cup D)$ be a triangulated punctured K -curve, let $\Sigma = \Sigma(X, V \cup D)$, and let $\varphi : \Sigma' \rightarrow \Sigma$ be a tame harmonic morphism of metrized complexes of curves. Then there exists a triangulated punctured K -curve $(X', V' \cup D')$ and a finite morphism $\psi : (X', V' \cup D') \rightarrow (X, V \cup D)$ such that the induced harmonic morphism of metrized complexes of curves is isomorphic to φ .*

Proof. For any finite vertex x' of Σ' at which φ , seen as a morphism of augmented metric graphs, is ramified, let $q_1, \dots, q_r \in C_v$ be all the branch points of $\varphi_{x'}$ which do not correspond to any edge of Σ , and let p_{ij} denote the preimages of q_i under $\varphi_{x'}$. We modify Σ' and Σ by attaching infinite edges e_i to Σ at $x = \varphi(x')$ for each q_i , infinite edges e'_{ij} to Σ' at x' for each p_{ij} , and defining $\text{red}_x(e_i) = q_i$ and $\text{red}_{x'}(e'_{ij}) = p_{ij}$. The harmonic morphism φ naturally extends to a tame covering $\tilde{\varphi} : \tilde{\Sigma}' \rightarrow \tilde{\Sigma}$ between the resulting modifications. Enlarging D to \tilde{D} by choosing points in $X(K) \setminus D$ with reduction q_i , we can assume that $\tilde{\Sigma} = \Sigma(X, V \cup \tilde{D})$. The result now follows from Theorem 1.4 by first lifting $\tilde{\varphi}$ to a tame covering $\psi : (X', V' \cup \tilde{D}') \rightarrow (X, V \cup \tilde{D})$ and then taking the restriction of ψ to $(X', V' \cup D') \rightarrow (X, V \cup D)$, where $D' = \psi^{-1}(D)$. ■

8. LIFTING HARMONIC MORPHISMS OF METRIC GRAPHS TO MORPHISMS OF METRIZED COMPLEXES

There is an obvious forgetful functor which sends metrized complexes of curves to (augmented) metric graphs, and morphisms of metrized complexes to morphisms of (augmented) metric graphs. A morphism of (augmented) metric graphs is said to be *liftable to a morphism of metrized complexes of k -curves* if it lies in the image of the forgetful functor.

We proved in Theorem 1.4 that every tame covering of metrized complexes of curves can be lifted to a tame covering of algebraic curves. In this section we study the problem of lifting harmonic morphisms of (augmented) metric graphs to finite morphisms of metrized complexes (and thus to tame coverings of proper smooth curves, thanks to Proposition 7.13).

8.1. Lifting finite augmented morphisms. Recall that k is an algebraically closed field of characteristic $p \geq 0$. A finite harmonic morphism φ of (augmented) metric graphs is called a *tame harmonic*

morphism if either $p = 0$ or all the local degrees along edges of φ are prime to p . Lifting of tame harmonic morphisms of augmented metric graphs to tame harmonic morphisms of metrized complexes of k -curves is equivalent to the existence of tamely ramified covers of k -curves of given genus with some given prescribed ramification profile.

8.1.1. A partition μ of an integer d is a multiset of natural numbers $d_1, \dots, d_l \geq 1$ with $\sum_i d_i = d$. The integer l , called the length of μ , will be denoted by $l(\mu)$.

Let $g', g \geq 0$ and $d > 0$ be integers, and let $M = \{\mu_1, \dots, \mu_s\}$ be a collection of s partitions of d . Assume that the integer R defined by

$$(8.1.2) \quad R := d(2 - 2g) + 2g' - 2 - sd + \sum_{i=1}^s l(\mu_i)$$

is non-negative. Denote by $\mathcal{A}_{g',g}^d(\mu_1, \dots, \mu_s)$ the set of all tame coverings $\varphi : C' \rightarrow C$ of smooth proper curves over k , with the following properties:

- (i) The curves C and C' are irreducible of genus g and g' , respectively;
- (ii) The degree of φ is equal to d ;
- (iii) The branch locus of φ contains (at least) s distinct points $x_1, \dots, x_s \in C$, and the ramification profile of φ at the points $\varphi^{-1}(x_i)$ is given by μ_i , for $1 \leq i \leq s$.

As we will explain now, the lifting problem for morphisms of augmented metric graphs to morphisms of metrized complexes over a field k reduces to the emptiness or non-emptiness of certain sets $\mathcal{A}_{g',g}^d(\mu_1, \dots, \mu_s)$. This latter problem is quite subtle, and no complete satisfactory answer is yet known (see also (8.3.1)). In some simple cases, however, one can ensure that $\mathcal{A}_{g',g}^d(\mu_1, \dots, \mu_s)$ is non-empty. For example, if all the partitions μ_i are trivial (i.e., they each consist of d 1's), then $\mathcal{A}_{g',g}^d(\mu_1, \dots, \mu_s)$ is non-empty. Here is another simple example.

Example 8.2. For an integer d prime to characteristic p of k , the set $\mathcal{A}_{0,0}^d((d), (d))$ is non-empty since it contains the map $z \mapsto z^d$. This is in fact the only map in $\mathcal{A}_{0,0}^d((d), (d))$ up to the action of the group $\mathrm{PGL}(2, k)$ on the target curve and \mathbf{P}^1 -isomorphisms of coverings.

8.2.1. Let $\varphi : \Gamma' \rightarrow \Gamma$ be a finite harmonic morphism of augmented metric graphs. Using the definition of a harmonic morphism, one can associate to any point p' of Γ' a collection $\mu_1(p'), \dots, \mu_s(p')$ of s partitions of the integer $d_{p'}(\varphi)$, where $s = \mathrm{val}(\varphi(p'))$, as follows: if $T_{\varphi(p)}(\Gamma) = \{v_1, \dots, v_s\}$ denotes all the tangent directions to Γ at $\varphi(p')$, then $\mu_i(p')$ is the partition of $d_{p'}(\varphi)$ which consists of the various local degrees of φ in all tangent directions $v' \in T_{p'}(\Gamma')$ mapping to v_i .

The next proposition is an immediate consequence of the various definitions involved once we note that, by Example 8.2, there are only finitely points $p' \in \Gamma'$ for which the question of non-emptiness of the sets $\mathcal{A}_{g(p'),g(\varphi(p'))}^{d_{p'}(\varphi)}$ arises. It provides a “numerical criterion” for a tame harmonic morphism of augmented metric graphs to be liftable to a tame harmonic morphism of metrized complexes of curves.

Proposition 8.3. *Let $\varphi : \Gamma' \rightarrow \Gamma$ be a tame harmonic morphism of augmented metric graphs. Then φ can be lifted to a tame harmonic morphism of metrized complexes over k if and only if for every point p' in Γ' , the set $\mathcal{A}_{g(p'),g(\varphi(p'))}^{d_{p'}(\varphi)}(\mu_1(p'), \dots, \mu_{\mathrm{val}(\varphi(p'))}(p'))$ is non-empty.*

8.3.1. In characteristic 0, the lifting problem for finite augmented morphisms of metric graphs can be further reduced to a vanishing question for certain Hurwitz numbers.

Fix an irreducible smooth proper curve C of genus g over k , and let $x_1, \dots, x_s, y_1, \dots, y_R$ be a set of distinct points on C . The Hurwitz set $\mathcal{H}_{g',g}^d(\mu_1, \dots, \mu_s)$ is the set of C -isomorphism classes of all coverings in $\mathcal{A}_{g',g}^d(\mu_1, \dots, \mu_s)$ satisfying (i), (ii), and (iii) in (8.1.1) for the curve C and the points x_1, \dots, x_s that we have fixed, and which in addition satisfy:

- (iv) The integer R is given by (8.1.2), and for each $1 \leq i \leq R$, φ has a unique simple ramification point y'_i lying above y_i .

Note that by the above condition, the branch locus of φ consists precisely of the points x_i, y_j . The Hurwitz number $H_{g',g}^d(\mu_1, \dots, \mu_s)$ is defined as

$$H_{g',g}^d(\mu_1, \dots, \mu_s) := \sum_{\varphi \in \mathcal{H}_{g',g}^d(\mu_1, \dots, \mu_s)} \frac{1}{|\text{Aut}_C(\varphi)|},$$

and does not depend on the choice of C and the closed points $x_1, \dots, x_s, y_1, \dots, y_R$ in C .

Example 8.4. It is known, see for example [EKS84], that

$$H_{g,0}^2 = \frac{1}{2}, \quad H_{g,0}^3((3), \dots, (3)) > 0, \quad H_{0,0}^4((2,2), (2,2), (3,1)) = 0.$$

For the reader's convenience, and since we will use it several times in the sequel, we sketch a proof of the fact that $H_{0,0}^4((2,2), (2,2), (3,1)) = 0$. By the Riemann-Hurwitz formula and the Riemann Existence Theorem, $H_{0,0}^4((2,2), (2,2), (3,1)) \neq 0$ if and only if there exist elements $\sigma_1, \sigma_2, \sigma_3$ in the symmetric group \mathfrak{S}_4 having cycle decompositions of type $(2,2), (2,2), (3,1)$, respectively, such that $\sigma_1\sigma_2\sigma_3 = 1$ and such that the σ_i generate a transitive subgroup of \mathfrak{S}_4 . However, elementary group theory shows that the product $\sigma_1\sigma_2$ cannot be of type $(3,1)$ (the transitivity condition does not intervene here). For a proof which works in any characteristic, see Lemma 10.10 below.

All Hurwitz numbers can be theoretically computed, for example using Frobenius Formula (see [LZ04, Theorem A.1.9]). Nevertheless, the problem of understanding their vanishing is wide open. The above example shows that Hurwitz numbers in degree at most three are all positive, which is not the case in degree four. Some families of (non-)vanishing Hurwitz numbers are known (see Example 8.5). However, in general one has to explicitly compute a given Hurwitz number to decide if this latter vanishes or not. We refer the reader to [EKS84], [PP06], and [PP08], along with the references therein, for an account of what is known about this subject. We will use the vanishing of $H_{0,0}^4((2,2), (2,2), (3,1))$ in Section 10 to construct a 4-gonal augmented graph (see Section 10 for the definition) which cannot be lifted to any 4-gonal proper smooth algebraic curve over K .

Example 8.5. Some partial results are known concerning the (non-)vanishing of Hurwitz numbers. For example, it is known that double Hurwitz numbers (i.e., when $s = 2$) are all positive (this can be seen for example from the presentation of the cut-join equation given in [CJM10]), as well as all the Hurwitz numbers $H_{g',g}^d(\mu_1, \dots, \mu_s)$ when $g \geq 1$ and $R \geq 0$ ([Hus62, EKS84]). On the other hand, it is proved in [PP08] that

$$H_{0,0}^d\left((d-2,2), (2, \dots, 2), \left(\frac{d}{2}+1, 1, \dots, 1\right)\right) = 0 \quad \text{for all } d \geq 4 \text{ even.}$$

The non-emptiness of $\mathcal{A}_{g',g}^d(\mu_1, \dots, \mu_s)$ can be reduced to the non-emptiness of the Hurwitz set $\mathcal{H}_{g',g}^d(\mu_1, \dots, \mu_s)$.

Lemma 8.6. Suppose that k has characteristic 0. Then $\mathcal{A}_{g',g}^d(\mu_1, \dots, \mu_s)$ is non-empty if and only if $H_{g',g}^d(\mu_1, \dots, \mu_s) \neq 0$.

Proof. Since $\mathcal{H}_{g',g}^d(\mu_1, \dots, \mu_s)$ is a subset of $\mathcal{A}_{g',g}^d(\mu_1, \dots, \mu_s)$, obviously we only need to prove that if $\mathcal{A}_{g',g}^d(\mu_1, \dots, \mu_s) \neq \emptyset$, then the Hurwitz set is also non-empty. Let $\varphi : C' \rightarrow C$ be an element of $\mathcal{A}_{g',g}^d(\mu_1, \dots, \mu_s)$, branched over $x_i \in C$ with ramification profile μ_i for $i = 1, \dots, s$, and let z_1, \dots, z_t be all the other points in the branch locus of φ . Denote by ν_i the ramification profile of φ above the point z_i . Fix a closed point \star of $C \setminus \{x_1, \dots, x_s, z_1, \dots, z_t\}$. The étale fundamental group $\pi_1(C \setminus \{x_1, \dots, x_s, z_1, \dots, z_t\}, \star)$ is the profinite completion of the group generated by a system of generators $a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_{s+t}$ satisfying the relation $[a_1, b_1] \dots [a_g, b_g] c_1 \dots c_{s+t} = 1$, where $[a, b] = aba^{-1}b^{-1}$ (see [SGA1]). In addition, the data of φ is equivalent to the data of a surjective

morphism ρ from $\pi_1(C \setminus \{x_1, \dots, x_s, z_1, \dots, z_t\}, \star)$ to a transitive subgroup of the symmetric group \mathfrak{S}_d of degree d such that the partition μ_i (resp. ν_i) of d corresponds to the lengths of the cyclic permutations in the decomposition of $\rho(c_i)$ (resp. $\rho(c_{s+i})$) in \mathfrak{S}_d into products of cycles, for $1 \leq i \leq s$ (resp. $1 \leq i \leq t$). By Riemann-Hurwitz formula, we have $R = \sum_{i=1}^t (d - l(\nu_i))$.

Now note that each $\rho(c_{s+i})$ can be written as a product of $d - l(\nu_i)$ transpositions $\tau_i^1, \dots, \tau_i^{d-l(\nu_i)}$ in \mathfrak{S}_d , i.e., $\rho(c_{s+i}) = \tau_i^1 \dots \tau_i^{d-l(\nu_i)}$. Rename the set of R distinct points y_1, \dots, y_R of $C \setminus \{x_1, \dots, x_s, \star\}$ as $z_i^1, \dots, z_i^{d-l(\nu_i)}$ for $1 \leq i \leq t$.

The étale fundamental group $\pi_1(C \setminus \{x_1, \dots, x_s, z_1^1, \dots, z_1^{d-l(\nu_1)}, \dots, z_t^{d-l(\nu_t)}\}, \star)$ has, as a profinite group, a system of generators $a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_s, c_{s+1}^1, \dots, c_{s+1}^{d-l(\nu_1)}, \dots, c_{s+t}^{d-l(\nu_t)}$ verifying the relation

$$[a_1, b_1] \dots [a_g, b_g] c_1 \dots c_s c_{s+1}^1 \dots c_{s+1}^{d-l(\nu_1)} \dots c_{s+t}^1 \dots c_{s+t}^{d-l(\nu_t)} = 1,$$

and admits a surjective morphism to \mathfrak{S}_d which coincides with ρ on $a_1, b_1, \dots, a_g, b_g$, and which sends c_{s+i}^j to τ_i^j for each $1 \leq i \leq t$ and $1 \leq j \leq d - l(\nu_i)$. The corresponding cover $C'' \rightarrow C$ obviously belongs to $\mathcal{A}_{g',g}^d(\mu_1, \dots, \mu_s)$ and in addition has simple ramification profile (2) above each y_i , i.e., it verifies condition (iv) above. This shows that $\mathcal{H}_{g',g}^d(\mu_1, \dots, \mu_s)$ is non-empty. \blacksquare

Corollary 8.7. *Suppose that k has characteristic 0. Let $\varphi : \Gamma' \rightarrow \Gamma$ be a finite morphism of augmented metric graphs, and let \mathcal{C} be a metrized complex over k lifting Γ . There exists a lifting of φ to a finite harmonic morphism of metrized complexes $\mathcal{C}' \rightarrow \mathcal{C}$ over k (and thus to a morphism of smooth proper curves over K) if and only if*

$$\prod_{p' \in V(\Gamma')} H_{g(p'), g(\varphi(p'))}^{d_{p'}(\varphi)}(\mu_1(p'), \dots, \mu_{\text{val}(\varphi(p'))}) \neq 0.$$

In particular, if φ is effective and $g(p) \geq 1$ for all the points of valency at least three in Γ , then φ lifts to a finite harmonic morphism of metrized complexes over k .

8.8. Lifting finite harmonic morphisms. Now we turn to the lifting problem for finite morphisms of non-augmented metric graphs to morphisms of metrized complexes of k -curves. In this case there are no obstructions to the existence of such a lift.

Theorem 8.9. *Let $\varphi : \Gamma' \rightarrow \Gamma$ be a tame harmonic morphism of metric graphs, and suppose that Γ is augmented. There exists an enrichment of Γ' to an augmented metric graph (Γ', g') such that $\varphi : (\Gamma', g') \rightarrow (\Gamma, g)$ lifts to a tame harmonic morphism of metrized complexes of curves over k (and thus to a morphism of smooth proper curves over K).*

Theorem 8.9 is an immediate consequence of Proposition 8.3 and the following theorem. (For the statement, we say that a partition μ of d is *tame* if either $\text{char}(k) = 0$ or all the integers appearing in μ are prime to p .)

Theorem 8.10. *Let $g \geq 0, d \geq 2, s \geq 1$ be integers. Let μ_1, \dots, μ_s be a collection of s tame partitions of d . Then there exists a sufficiently large non-negative integer g' such that $\mathcal{A}_{g',g}^d(\mu_1, \dots, \mu_s)$ is non-empty.*

Proof. We first give a simple proof which works in characteristic zero, and more generally, in the case of a tame monodromy group. The proof in characteristic $p > 0$ is based on our lifting theorem and a deformation argument.

Suppose first that the characteristic of k is zero. By Lemma 8.6, we need to show that for large enough g' the set $\mathcal{H}_{g',g}^d(\mu_1, \dots, \mu_s)$ is non-empty.

If $g \geq 1$, for any large enough g' giving $R \geq 0$, we have $\mathcal{H}_{g',g}^d(\mu_1, \dots, \mu_s) \neq \emptyset$ [Hus62]. So suppose $g = 0$. Consider $s + R + 1$ distinct points $x_1, \dots, x_s, z_1, \dots, z_R, \star$ in C . The étale fundamental group $\pi_1(R) := \pi_1(C \setminus \{x_1, \dots, x_s, z_1, \dots, z_R\}, \star)$ has, as a profinite group, a system of generators $c_1, \dots, c_s, c_{s+1}, \dots, c_{s+R}$ verifying the relation

$$c_1 \dots c_s c_{s+1} \dots c_{s+R} = 1.$$

It will be enough to show that for a large enough R , there exists a surjective morphism ρ from $\pi_1(R)$ to \mathfrak{S}_d so that $\rho(c_{s+i})$ is a transposition for any $i = 1, \dots, R$, and that for any $i = 1, \dots, s$, the partition of d given by the lengths of the cyclic permutations in the decomposition of $\rho(c_i)$ is equal to μ_i . In this case, the genus g' of the corresponding cover C' of C in $\mathcal{H}_{g',0}^d(\mu_1, \dots, \mu_s)$ will be given by

$$g' = 1 - d + \frac{1}{2} \left[sd + R - \sum_{i=1}^s l(\mu_i) \right].$$

Consider an arbitrary map ρ from $\{c_1, \dots, c_s\}$ to \mathfrak{S}_d verifying the ramification profile condition for $\rho(c_1), \dots, \rho(c_s)$. Choose a system of d transpositions τ_1, \dots, τ_d generating \mathfrak{S}_d , and consider a set of transpositions $\tau_{d+1}, \dots, \tau_R$ such that

$$\rho(c_1) \dots \rho(c_s) \tau_1 \dots \tau_d = \tau_R \dots \tau_{d+1}.$$

This proves Theorem 8.10 when k has characteristic 0.

Consider now the case of a base field k of positive characteristic $p > 0$. Note that since the prime to p part of the tame fundamental group has the same representation as in the case of characteristic zero, the group theoretic method we used in the previous case can be applied if the monodromy group is tame, i.e., has size prime to p . However, in general it is impossible to impose such a condition on the monodromy group. For example in the case when p divides d , the size of the monodromy group is always divisible by p .

We first describe how to reduce the proof of Theorem 8.10 to the case $s = 1$ and $g = 0$. Suppose that for each μ_i , $1 \leq i \leq s$, there exists a large enough g_i such that $\mathcal{A}_{g_i,0}^d(\mu_i)$ is non-empty, and consider a tame cover $\varphi_i : C_i \rightarrow \mathbf{P}_k^1$ in $\mathcal{A}_{g_i,0}^d(\mu_i)$ such that the ramification profile over $0 \in \mathbf{P}^1$ is given by μ_i , and choose two regular points $x_i, y_i \in \mathbf{P}^1$ (i.e. x_i, y_i are outside the branch locus of φ_i). Choose also a smooth proper curve C_0 of genus g which admits a tame cover $\varphi_0 : C'_0 \rightarrow C_0$ of degree d from a smooth proper curve C'_0 of large enough genus g'_0 . (The existence of such a cover can be deduced by a similar trick as that discussed at the end of the proof below and depicted in Figure 9.) Let $y_0 \in C_0$ be a regular point of φ_0 .

Let \mathcal{C}_0 be the metrized complex over k whose underlying metric graph is $[0, +\infty]$, with one finite vertex v_0 and one infinite vertex v_∞ , equipped with the metric induced by \mathbf{R} , and with $C_{v_0} = C_0$ and $\text{red}_{v_0}(\{v_0, v_\infty\}) = y_0$. Denote by \mathcal{C} the modification of \mathcal{C}_0 obtained by taking a refinement at r distinct points $0 < v_1 < \dots < v_s < \infty$, as depicted in Figure 8, and by setting $C_{v_i} = \mathbf{P}^1$ and $\text{red}_{v_i}(\{v_i, v_{i-1}\}) = x_i$ and $\text{red}_{v_i}(\{v_i, v_{i+1}\}) = y_i$ (here $v_{s+1} = v_\infty$), and by adding an infinite edge e_i to each v_i , and defining $\text{red}_{v_i}(e_i) = 0 \in \mathbf{P}^1$. Denote by Γ the underlying metric graph of \mathcal{C} . See Figure 8.

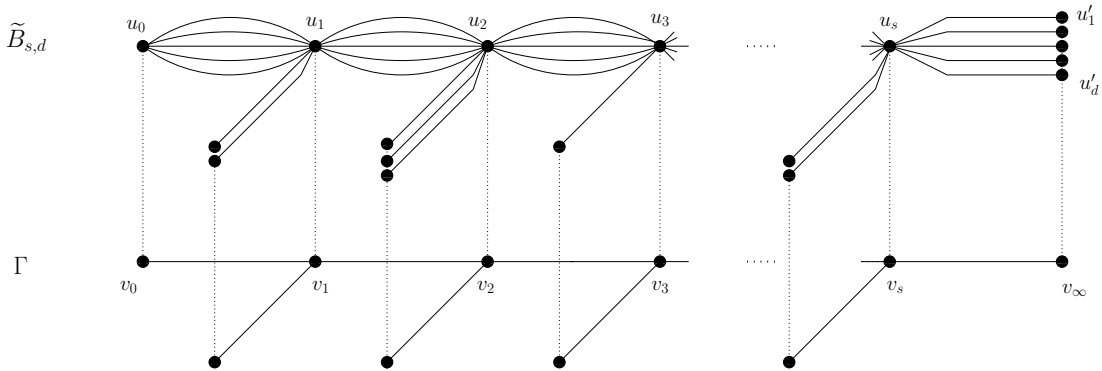


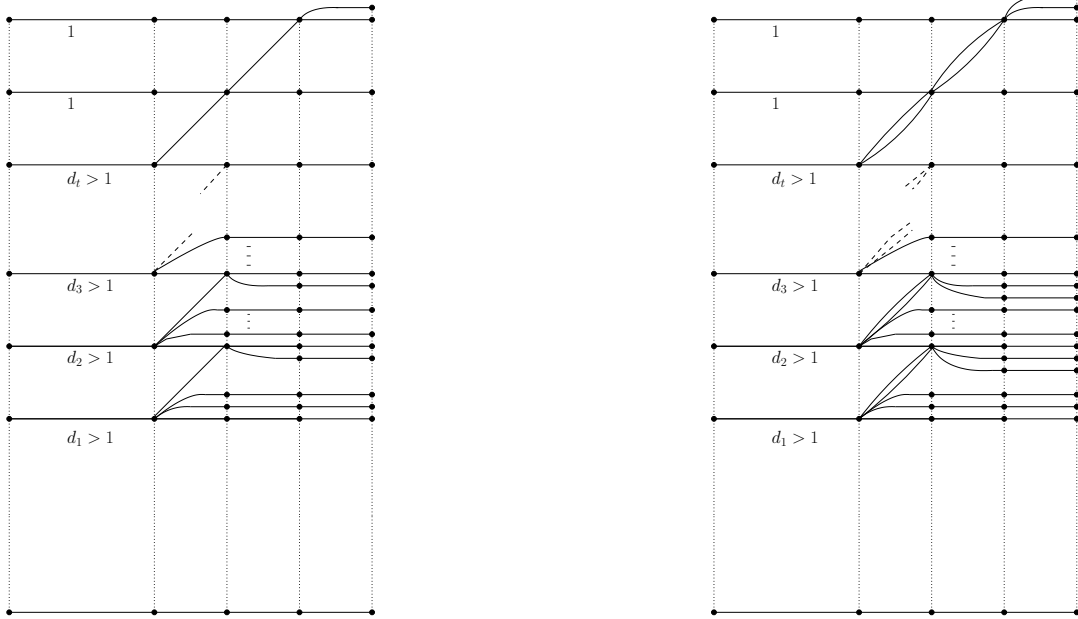
FIGURE 8.

Define now the metric graph $B_{s,d}$ as the chain of s banana graphs of size d : $B_{s,d}$ has $s + 1$ finite vertices u_0, \dots, u_s and u'_1, \dots, u'_d infinite vertices adjacent to u_s such that u_i is connected to u_{i+1} with

d edges of length $\ell_\Gamma(\{v_{i+1} - v_i\})$. We denote by $\tilde{B}_{s,d}$ the tropical modification of $B_{r,d}$ at u_1, \dots, u_s , obtained by adding $l(\mu_i)$ infinite edges to u_i . Eventually we turn $\tilde{B}_{s,d}$ into a metrized complex $\mathcal{C}_{s,d}$ over k by setting $C_{u_i} = C_i$, and defining red_{u_i} on the d edges between u_i and u_{i+1} by a bijection to the d points in $\varphi_i^{-1}(y_i)$, red_{u_i} on the edges between u_i and u_{i-1} by a bijection to the d points in $\varphi_i^{-1}(x_i)$, and red_{u_i} on the $l(\mu_i)$ infinite edges adjacent to u_i by a bijection to the $l(\mu_i)$ points in $\varphi_i^{-1}(0)$.

Obviously, there exists a degree d tame morphism $\varphi : \mathcal{C}_{s,d} \rightarrow \mathcal{C}$ of curve complexes over k which sends u_i to v_i , and has degrees given by integers in μ_i above the infinite edge of Γ adjacent to v_i , for $i = 1, \dots, s$, and $\varphi_{u_i} = \varphi_i$ (see Figure 8). According to Proposition 7.13, the map φ lifts to a tame morphism of smooth proper curves $\varphi_K : X \rightarrow X'$ over K the completion of the algebraic closure of $k[[t]]$. The map φ_K has partial ramification profile μ_1, \dots, μ_s . To deduce now the non-emptiness of $\mathcal{A}_{g',g}^d(\mu_1, \dots, \mu_s)$, we note that there exists a subring R of K finitely presented over k such that the map φ_K descends to a finite morphism $\varphi_R : \mathfrak{X} \rightarrow \mathfrak{X}'$ between smooth proper curves over $\text{Spec}(R)$. In addition, over a non-empty open subset U of $\text{Spec}(R)$, φ_R specializes to a tame cover with the same ramification profile as φ_K . Since U contains a k -rational point, we infer the existence of a large enough g' such that $\mathcal{A}_{g',g}^d(\mu_1, \dots, \mu_s) \neq \emptyset$.

We are thus led to consider the case where $s = 1$, $g = 0$, $\mu = (d_1, d_2, \dots, d_l)$ with $\sum_i d_i = d$, $d_1, \dots, d_l > 1$, and $d_{t+1} = \dots = d_l = 1$. Figure 9 shows that, just as in the previous reduction, one can reduce to the case where $s = 1$ and $\mu_1 = \{d\}$ with $(d, p) = 1$. But this is just non-emptiness of $\mathcal{A}_{0,0}((d))$ (see Example 8.2). \blacksquare



a) Reduction in the case $p \neq 2$.

b) Reduction in the case $p = 2$.

FIGURE 9.

Remark 8.11. As the above proof shows, when k has characteristic zero one can get an explicit upper bound on the smallest positive integer g' with $\mathcal{H}_{g',0}^d(\mu_1, \dots, \mu_s) \neq \emptyset$. Indeed, the permutation $\rho(c_1) \dots \rho(c_s) \tau_1 \dots \tau_d$ can be written as the product of $d + \sum_{i=1}^s (d - l(\mu_i))$ transpositions. So without loss of generality we have $R - d = d + \sum_{i=1}^s (d - l(\mu_i))$, which means that one can take g' to be

$1 + \sum_{i=1}^r (d - l(\mu_i))$. For $g \geq 1$, $\mathcal{H}_{g',g}$ is non-empty as soon as R is non-negative, which means in this case that one can take g' to be $1 + (g - 1)d + \frac{1}{2} \sum_i (d - l(\mu_i))$.

8.12. Lifting of harmonic morphisms in the case the base has genus zero. We now consider the special case where Γ has genus zero and present more refined lifting results in this case. As explained in (2.27), a given harmonic morphism of (augmented) metric graphs does not necessarily have a tropical modification which is finite. We present below a weakened notion of finiteness of a harmonic morphism, and prove that any harmonic morphism from an (augmented) metric graph to an (augmented) rational metric graph satisfies this weak finiteness property. We discuss in Section 9 some consequences concerning linear equivalence of divisors on metric graphs.

Definition 8.13. A harmonic morphism $\varphi : \Gamma \rightarrow T$ from an augmented metric graph Γ to a metric tree T is said to admit a *weak resolution* if there exists a tropical modification $\tau : \tilde{\Gamma} \rightarrow \Gamma$ and an augmented harmonic morphism $\tilde{\varphi} : \tilde{\Gamma} \rightarrow T$ such that the restriction $\tilde{\varphi}|_{\Gamma}$ is equal to φ , and some tropical modification of $\tilde{\varphi}$ is finite.

In other words, the morphism φ has a weak resolution if it can be extended, up to increasing the degree of φ using the modification τ , to a tropical morphism $\tilde{\varphi} : \tilde{\Gamma} \rightarrow T$.

Example 8.14. The harmonic morphism depicted in Figure 2(d) with $d = 1$ can be weakly resolved by the harmonic morphisms depicted in Figures 5(a) and 2(c). Another example of a weak resolution is depicted in Figure 10.

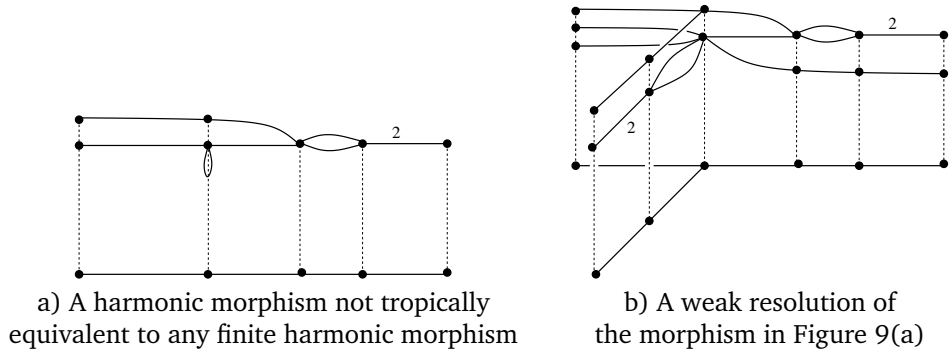


FIGURE 10.

Definition 8.15. Let $\varphi : \Gamma \rightarrow T$ be a harmonic morphism from a metric graph Γ to a metric tree T . A point $p \in \Gamma$ is *regular* if there exists a neighborhood U of p in Γ such that φ is not constant on U .

The *contracted set* of φ , denoted by $\mathcal{E}(\varphi)$, is the set of all non-regular points of φ . A *contracted component* of φ is a connected component of $\mathcal{E}(\varphi)$.

The next proposition, together with Proposition 7.13, allows us to conclude that any harmonic morphism from an augmented metric graph to a metric tree can be realized, up to weak resolutions, as the induced morphism on skeleta of a finite morphism of triangulated punctured curves. Recall that $\Lambda = \text{val}(K^\times)$ is divisible since K is algebraically closed.

Proposition 8.16. (Weak resolution of contractions) *Let $\varphi : \Gamma \rightarrow T$ be a harmonic morphism of degree d from a metric graph Γ to a metric tree T .*

- (1) *There exist tropical modifications $\tau : \tilde{\Gamma} \rightarrow \Gamma$ and $\tau' : \tilde{T} \rightarrow T$, and a harmonic morphism of metric graphs (of degree $\tilde{d} \geq d$) $\tilde{\varphi} : \tilde{\Gamma} \rightarrow \tilde{T}$, such that $\tilde{\varphi}|_{\Gamma \setminus \mathcal{E}(\varphi)} = \varphi$, where $\mathcal{E}(\varphi)$ is the contracted part of Γ .*
- (2) *Suppose in addition that Γ is augmented, and if $p > 0$ that all the non-zero degrees of φ along tangent directions at Γ are prime to p . Then there exist tropical modifications of Γ , T , and φ as above such that $\tilde{\varphi}$ is tame and, in addition, there exists a tame harmonic morphism of metrized*

complexes of k -curves with $\tilde{\varphi}$ as the underlying finite harmonic morphism of augmented metric graphs.

Proof. Up to tropical modifications, we may assume that all 1-valent vertices of T are infinite vertices.

The proof of (1) goes by giving an algorithm to exhibit a weak resolution of φ . Note that this algorithm does not produce the weak resolutions presented in Example 8.14, since in these cases we could find simpler ones.

Let $V(\Gamma)$ be any strongly semi-stable vertex set of Γ . We denote by d the degree of φ , and by α the number of non-regular vertices of φ . Given v a non-regular vertex of Γ , we consider the tropical modification $\tau_v : \tilde{\Gamma}_v \rightarrow \Gamma$ such that $(\tilde{\Gamma}_v \setminus \Gamma) \cup \{v\}$ is isomorphic to T as a metric graph. Considering all those modifications for all non-regular vertices of φ , we obtain a modification $\tau : \tilde{\Gamma} \rightarrow \Gamma$. We can naturally extend φ to a harmonic morphism $\tilde{\varphi} : \tilde{\Gamma} \rightarrow T$ of degree $d + \alpha$ such that $\tilde{\varphi}|_{\Gamma} = \varphi$ and all degrees of $\tilde{\varphi}$ on edges not in Γ are equal to 1 (see Figure 11(a) in the case of the harmonic morphism depicted in Figure 2(d) with $d = 1$).

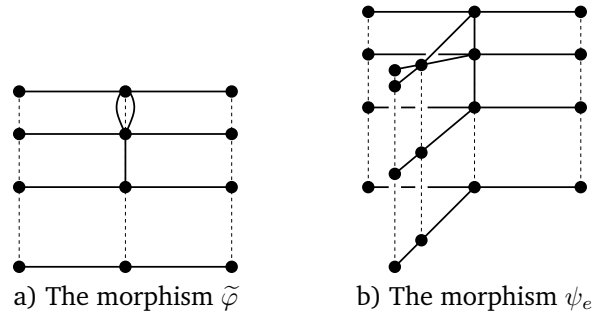


FIGURE 11. The harmonic morphisms $\tilde{\varphi}$ and ψ_e in the case of Figure 2(d) with $d = 1$

By construction, any contracted component of $\tilde{\varphi}$ now reduces to an edge of Γ , and this can be easily resolved. Indeed, if e is a finite contracted edge of $\tilde{\varphi}$, we do the following (see Figure 11(b)):

- consider the tropical modification $\tau_T : \tilde{T} \rightarrow T$ of T at $\tilde{\varphi}(e)$; denote by e_1 the new end of \tilde{T} ;
- consider $\tau_e : \tilde{\Gamma}_e \rightarrow \tilde{\Gamma}$ the composition of two elementary tropical modifications of $\tilde{\Gamma}$ at the middle of the edge e ; denote by e_2 and e_3 the two new infinite edges of $\tilde{\Gamma}_e$, and by e_4 and e_5 the two new finite edges of $\tilde{\Gamma}_e$;
- consider the morphism of metric graphs $\tilde{\varphi}_e : \tilde{\Gamma}_e \rightarrow \tilde{T}$ defined by
 - $\tilde{\varphi}_e|_{\tilde{\Gamma}_e \setminus \{e_2, e_3, e_4, e_5\}} = \tilde{\varphi}$,
 - $\tilde{\varphi}_e(e_2 \cup e_3 \cup e_4 \cup e_5) = e_1$,
 - $d_{e_i}(\tilde{\varphi}_e) = 1$ for $i = 2, 3, 4, 5$.
- extend $\tilde{\varphi}_e$ to a harmonic morphism of metric graphs $\psi_e : \Gamma' \rightarrow \tilde{T}$, where Γ' is a modification of $\tilde{\Gamma}_e$ at regular vertices in $\tilde{\varphi}_e^{-1}(\tilde{\varphi}(e))$, with all degrees of $\tilde{\varphi}$ on edges not in $\tilde{\Gamma}_e$ equal to 1.

We resolve in the same way a contracted infinite end of $\tilde{\Gamma}$. By applying this process to all contracted edges of $\tilde{\varphi}$, we end up with a finite harmonic morphism of metric graphs which is a tropical modification of $\tilde{\varphi}$.

The proof of (2) follows the same steps as the proof of (1), using in addition the following claim.

Claim. Let $g' \geq 0$ and $d, s > 0$ be integers. Let μ_1, \dots, μ_s be a collection of s tame partitions of d . Then there exist arbitrarily large non-negative integers d' such that $\mathcal{A}_{g',0}^{d'}(\mu'_1, \dots, \mu'_s)$ is non-empty, where μ'_i is the partition of d' obtained by adding a sequence of $d' - d$ numbers 1 to each partition μ_i .

Figure 12, Figure 9(a), our resolution procedure, and the argument used for the positive characteristic case of the proof of Theorem 8.10 reduce the proof of the claim to the case $s = 1$ and $\mu_1 = \{d\}$ with $(d, p) = 1$. But in this case, for any $g' \geq 0$, by the group theoretic method we used in the proof of Theorem 8.10, there exists a (tame) covering of \mathbf{P}^1 by a curve of genus g' having (tame) monodromy group the cyclic group $\mathbf{Z}/d\mathbf{Z}$, and with the property that the ramification profile above the point 1 of \mathbf{P}^1 is given by $\mu = \{d\}$. This finishes the proof of the claim, and the proposition follows. ■

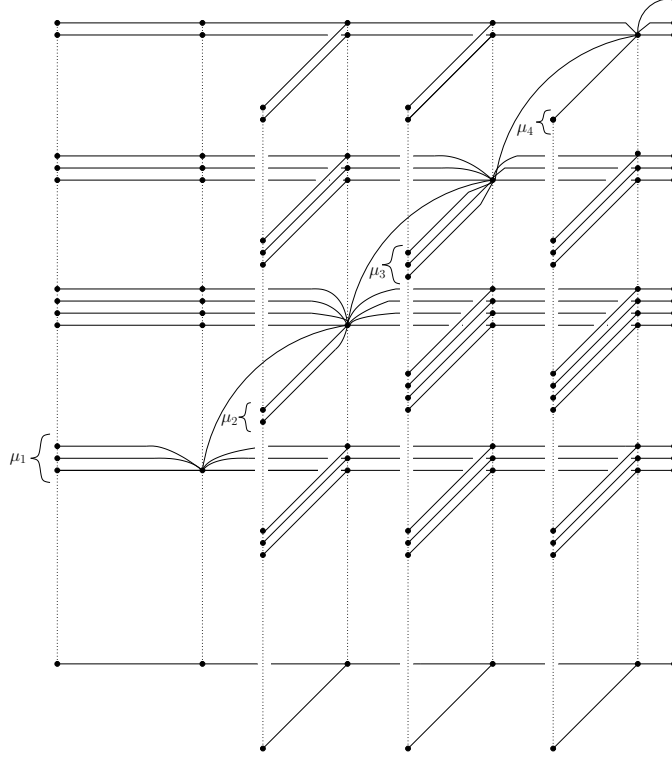


FIGURE 12. Reduction to the case $r = 1$ in the proof of (2) in Proposition 8.16. Degrees on (infinite) edges related to μ_i are exactly the integers appearing in μ_i . All the other degrees are one. Degrees over each infinite edge consist of a μ_i and precisely $(r - 1)d$ numbers 1.

9. APPLICATIONS

9.1. Harmonic morphisms of \mathbf{Z} -metric graphs and component groups of Néron models. In contrast to the rest of the paper, we assume in this section that R is a complete *discrete valuation ring* with fraction field K and algebraically closed residue field k . In this case we take the value group Λ to be \mathbf{Z} .

There is a natural notion of harmonic 1-forms on a metric graph Γ of genus g (see [MZ08]). The space $\Omega^1(\Gamma)$ is a g -dimensional real vector space which can be canonically identified with $H^1(\Gamma, \mathbf{R})$, but we write elements of $\Omega^1(\Gamma)$ as $\omega = \sum_e \omega_e de$ as in [BF11], where the sum is over all edges of a fixed model G for Γ . There is a canonical lattice $\Omega_{\mathbf{Z}}^1(\Gamma)$ of *integer harmonic 1-forms* inside $\Omega^1(\Gamma)$; these are the harmonic 1-forms for which every ω_e is an integer. A harmonic morphism $\varphi : \Gamma' \rightarrow \Gamma$ induces a natural pullback map on harmonic 1-forms via the formula

$$\varphi^*\left(\sum_e \omega_e de\right) = \sum_{e'} m_{\varphi}(e') \cdot \omega_{\varphi(e')} de'.$$

Recall that a **Z**-metric graph (i.e. a Λ -metric graph for $\Lambda = \mathbf{Z}$) with no infinite vertices is a (compact and finite) metric graph having a model whose edge lengths are all positive integers (or equivalently, having a model all of whose edge lengths are 1). If X/K is a smooth, proper, geometrically connected analytic curve and \mathfrak{X} is a semistable R -model for X , the skeleton $\Gamma_{\mathfrak{X}}$ of \mathfrak{X} is naturally a **Z**-metric graph. Moreover, as we have seen earlier in this paper, a finite morphism of semistable models induces in a natural way a harmonic morphism of **Z**-metric graphs.

Let Γ be a **Z**-metric graph with no infinite vertices. We define the *regularized Jacobian* $\text{Jac}_{\text{reg}}(\Gamma)$ of Γ to be the group $\text{Jac}(G)$, where G is the “regular model” for Γ (the unique weighted graph model having all edge lengths equal to 1). The group $\text{Jac}(G)$ can be described explicitly as

$$\text{Jac}(G) = \Omega^1(G)^{\#} / H_1(G, \mathbf{Z}),$$

where $\Omega^1(G)^{\#}$ denotes the linear functionals $\Omega^1(G, \mathbf{R}) \rightarrow \mathbf{R}$ of the form \int_{α} with $\alpha \in C_1(G, \mathbf{Z})$ (see [BF11]). There is a canonical isomorphism between $\text{Jac}(G)$ and $\text{Pic}^0(G)$, the group of divisors of degree 0 on G modulo the principal divisors (see [BdlHN97]) as well as a canonical isomorphism

$$\text{Jac}(G) \cong H^1(G, \mathbf{Z})^{\#} / \Omega_{\mathbf{Z}}^1(G),$$

where

$$H^1(G, \mathbf{Z})^{\#} = \left\{ \omega \in \Omega^1(G, \mathbf{R}) : \int_{\gamma} \omega \in \mathbf{Z} \forall \gamma \in H_1(G, \mathbf{Z}) \right\}.$$

We recall (in our own terminology) the following result of Raynaud (cf. [Ray70] and [Bak08, Appendix A]):

Theorem 9.2. (Raynaud) *If X/K is a semistable curve, then the component group of the Néron model of $\text{Jac}(X)$ over R is canonically isomorphic to $\text{Jac}_{\text{reg}}(\Gamma_{\mathfrak{X}})$ (for any semistable model \mathfrak{X} of X).*

A harmonic morphism $\varphi : \Gamma' \rightarrow \Gamma$ of **Z**-metric graphs induces in a functorial way homomorphisms $\varphi_* : \text{Jac}_{\text{reg}}(\Gamma') \rightarrow \text{Jac}_{\text{reg}}(\Gamma)$, defined by

$$\varphi_* \left(\left[\int_{\alpha'} \right] \right) = \left[\int_{\varphi_*(\alpha')} \right],$$

where

$$\varphi_* \left(\sum \alpha'_{e'} e' \right) = \sum \alpha_{e'} \varphi(e'),$$

and $\varphi^* : \text{Jac}_{\text{reg}}(\Gamma) \rightarrow \text{Jac}_{\text{reg}}(\Gamma')$, defined by

$$\varphi^*([\omega]) = [\varphi^* \omega],$$

where

$$\varphi^* \left(\sum \omega_e de \right) = \sum_{e'} m_{\varphi}(e') \omega_{\varphi(e')} d e'.$$

We have the following elementary result, whose proof we omit:

Lemma 9.3. *Under the canonical isomorphism between $\text{Jac}(G)$ and $\text{Pic}^0(G)$, the homomorphism $\varphi_* : \text{Jac}(G') \rightarrow \text{Jac}(G)$ corresponds to the map $[D'] \mapsto [\varphi_*(D')]$ from $\text{Pic}^0(G')$ to $\text{Pic}^0(G)$, where $\varphi_*(\sum a_{v'}(v')) = \sum a_{v'} \varphi(v')$.*

The following is a “relative” version of Raynaud’s theorem; it implies that the covariant functor which takes a semistable R -model \mathfrak{X} for a K -curve X to the component group of the Néron model of the Jacobian of X factors as the “reduction graph functor” $\mathfrak{X} \mapsto \Gamma_{\mathfrak{X}}$ followed by the “regularized Jacobian functor” $\Gamma \mapsto \text{Jac}_{\text{reg}}(\Gamma)$. It is a straightforward consequence of the analytic description of Raynaud’s theorem given in [Bak08, Appendix A] (see also [BR13]):

Theorem 9.4. *If $f_K : X' \rightarrow X$ is a finite morphism of curves over K , the induced maps $f_* : \Phi_{J(X')} \rightarrow \Phi_{J(X)}$ and $f^* : \Phi_{J(X)} \rightarrow \Phi_{J(X')}$ on component groups coincide with the induced maps $\varphi_* : \text{Jac}_{\text{reg}}(\Gamma_{\mathfrak{X}'}) \rightarrow \text{Jac}_{\text{reg}}(\Gamma_{\mathfrak{X}})$ and $\varphi^* : \text{Jac}_{\text{reg}}(\Gamma_{\mathfrak{X}}) \rightarrow \text{Jac}_{\text{reg}}(\Gamma_{\mathfrak{X}'})$ for any morphism $f : \mathfrak{X}' \rightarrow \mathfrak{X}$ of*

semistable models extending f_K . (Here φ denotes the harmonic morphism of skeleta induced by f ; see Remark 5.14.)

It follows easily from Lemma 9.3 and Theorem 9.4 that if $f_K : X' \rightarrow X$ is *regularizable*, i.e., if f_K extends to a morphism of regular semistable models, then the induced map $f_* : \Phi_{X'} \rightarrow \Phi_X$ on component groups is surjective. Thus whenever f_* is not surjective, it follows that f_K does not extend to a morphism of regular semistable models. One can obtain a number of concrete examples of this situation from modular curves (e.g. the map $X_0(33) \rightarrow E$ over $\mathbf{Q}_3^{\text{unr}}$, where E is the optimal elliptic curve of level 33.)

Remark 9.5. One can show that if $\varphi : \Gamma' \rightarrow \Gamma$ is a harmonic morphism of \mathbf{Z} -metric graphs, then $\varphi_* : \Phi_{X'} \rightarrow \Phi_X$ is surjective iff $\varphi^* : \Phi_X \rightarrow \Phi_{X'}$ is injective. Indeed, it is not hard to check that the maps φ_* and φ^* are adjoint with respect to the *combinatorial monodromy pairing*, the non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle : \text{Jac}(G) \times \text{Jac}(G) \rightarrow \mathbf{Q}/\mathbf{Z}$ defined by $\langle [\omega], [\int_\alpha] \rangle = [\int_\alpha \omega]$, where $\omega \in H_1(G, \mathbf{Z})^\#$ and $\int_\alpha \in \Omega^1(G)^\#$. Thus the groups $\ker(\varphi^*)$ and $\text{coker}(\varphi_*)$ are canonically dual. This is a combinatorial analogue of results proved by Grothendieck in SGA7 on the (usual) monodromy pairing.

As an application of Theorem 9.4 and our results on lifting harmonic morphisms, one can construct many examples of harmonic morphisms of \mathbf{Z} -metric graphs for which φ_* is not surjective. For example, consider the following question posed by Ken Ribet in a 2007 email correspondence with the second author (Baker):

Suppose $f : X' \rightarrow X$ is a finite morphism of semistable curves over a complete discretely valued field K with $g(X) \geq 2$. Assume that the special fiber of the minimal regular model of X' consists of two projective lines intersecting transversely. Is the induced map $f_* : \Phi_{X'} \rightarrow \Phi_X$ on component groups of Néron models necessarily surjective?

We now show that the answer to Ribet's question is **no**.

Example 9.6. Consider the “banana graph” $B(\ell_1, \dots, \ell_{g+1})$ consisting of two vertices and $g+1$ edges of length ℓ_i . This is the reduction graph of a semistable curve whose reduction has two \mathbf{P}^1 's crossing transversely at singular points of thickness $\ell_1, \dots, \ell_{g+1}$. If we set $G' = B(1, 1, 1, 1)$ and $G = B(1, 2, 2)$ and let Γ', Γ be the geometric realizations of G' and G , respectively, then there is a degree 2 harmonic morphism of \mathbf{Z} -metric graphs $\varphi : \Gamma' \rightarrow \Gamma$ taking e'_1 and e'_2 to e_1 , e'_3 to e_2 , and e'_4 to e_3 . The homomorphism φ_* is non-surjective since $|\text{Jac}_{\text{reg}}(\Gamma)| = 4$ and $|\text{Jac}_{\text{reg}}(\Gamma')| = 8$. The map φ can be enriched to a harmonic morphism $\tilde{\varphi}$ of metrized complexes of curves by attaching a \mathbf{P}^1 to each vertex and letting each morphism $\mathbf{P}^1 \rightarrow \mathbf{P}^1$ be $z \mapsto z^2$, with $0, \pm 1$, and ∞ being the marked points upstairs. By Remark 7.11, the morphism $\tilde{\varphi}$ lifts to a morphism $\psi : X' \rightarrow X$ of curves over K . Since all edges of G have length 1, the minimal proper regular model of X' has a special fiber consisting of two \mathbf{P}^1 's crossing transversely. By Theorem 9.4, the induced covariant map on component groups of Néron models is not surjective.

9.7. Linear equivalence of divisors. A (*tropical*) *rational function* on a metric graph Γ is a continuous piecewise affine function $F : \Gamma \rightarrow \mathbf{R}$ with integer slopes. If F is a rational function on Γ , $\text{div}(F)$ is the divisor on Γ whose coefficient at a point x of Γ is given by $\sum_{v \in T_x} d_v F$, where the sum is over all tangent directions to Γ at x and $d_v F$ is the outgoing slope of F at x in the direction v . Two divisors D and D' on a metric graph Γ are called *linearly equivalent* if there exists a rational function F on Γ such that $D - D' = \text{div}(F)$, in which case we write $D \sim D'$. For a divisor D on Γ , the *complete linear system of D* , denoted $|D|$, is the set of all effective divisors E linearly equivalent to D . The *rank* of a divisor $D \in \text{Div}(\Gamma)$ is defined to be

$$r_\Gamma(D) := \min_{\substack{E : E \geq 0 \\ |D - E| = \emptyset}} \deg(E) - 1.$$

Let $\varphi : \Gamma \rightarrow T$ be a finite harmonic morphism from Γ to a metric tree T of degree d . For any point $x \in T$, the (local degree of φ at the points of the) fiber $\varphi^{-1}(x)$ defines a divisor of degree d in $\text{Div}(\Gamma)$ that we denote by $D_x(\varphi)$. We have

$$D_x(\varphi) := \sum_{y \in \varphi^{-1}(x)} d_y(\varphi)(y),$$

where $d_y(\varphi)$ denotes the local degree of φ at y .

Proposition 9.8. *Let $\varphi : \Gamma \rightarrow T$ be a finite harmonic morphism of degree d from Γ to a metric tree. Then for any two points x_1 and x_2 in T , we have $D_{x_1}(\varphi) \sim D_{x_2}(\varphi)$ in Γ . Moreover, for every $x \in T$ the rank of the divisor $D_x(\varphi)$ is at least one.*

Proof. Since T is connected, we may assume that x_1 and x_2 are sufficiently close; more precisely, we can suppose that x_2 lies on the same edge as x_1 with respect to some model G for Γ . Removing the open segment (x_1, x_2) from T leaves two connected components T_{x_1} and T_{x_2} which contain x_1 and x_2 , respectively. Identifying the segment $[x_1, x_2]$ with the interval $[0, \ell]$ by a linear map (where $\ell = \ell([x_1, x_2])$ denotes the length in T of the segment $[x_1, x_2]$) gives a rational function $F : \Gamma \rightarrow [0, \ell]$ by sending $\varphi^{-1}(T_{x_1})$ and $\varphi^{-1}(T_{x_2})$ to 0 and ℓ , respectively. It is easy to verify that $D_{x_1}(\varphi) - D_{x_2}(\varphi) = \text{div}(F)$, which establishes the first part.

The second part follows from the first, since y belongs to the support of the divisor $D_{\varphi(y)}(\varphi) \sim D_x(\varphi)$ for all $y \in \Gamma$, which shows that $r_\Gamma(D_x(\varphi)) \geq 1$. \blacksquare

By Theorem 8.9, any finite morphism $\varphi : \Gamma \rightarrow T$ can be lifted to a morphism $\varphi : X \rightarrow \mathbf{P}^1$ of smooth proper curves, possibly with $g(X) > g(\Gamma)$. This shows that any effective divisor on Γ which appears as a fiber of a finite morphism to a metric tree can be lifted to a divisor of rank at least one on a smooth proper curve of possibly higher genus.

We are now going to show that the (additive) equivalence relation generated by fibers of “tropicalization” of finite morphisms $X \rightarrow \mathbf{P}^1$ coincides with tropical linear equivalence of divisors. To give a more precise statement, let Γ be a metric graph with first Betti number $h_1(\Gamma)$, and consider the family of all smooth proper curves of genus $h_1(\Gamma)$ over K which admit a semistable vertex set V and a finite set of K -points D such that the metric graph $\Sigma(X, V \cup D)$ is a modification of Γ . Given such a curve X and a finite morphism $\varphi : X \rightarrow \mathbf{P}^1$, there is a corresponding finite harmonic morphism $\varphi : \Sigma(X, V \cup D) \rightarrow T$ from a modification of Γ to a metric tree T . Two effective divisors D_0 and D_1 on Γ are called *strongly effectively linearly equivalent* if there exists a morphism $\varphi : \Sigma(X, V \cup D) \rightarrow T$ as above such that $D_0 = \tau_*(D_{x_0}(\varphi))$ and $D_1 = \tau_*(D_{x_1}(\varphi))$ for two points x_0 and x_1 in T . Here $\tau_* : \text{Div}(\Sigma(X, V \cup D)) \rightarrow \text{Div}(\Gamma)$ is the extension by linearity of the retraction map $\tau : \Sigma(X, V \cup D) \rightarrow \Gamma$. The equivalence relation on the abelian group $\text{Div}(\Gamma)$ generated by this relation is called *effective linear equivalence* of divisors. In other words, two divisors D_0 and D_1 on Γ are effectively linearly equivalent if and only if there exists an effective divisor E on Γ such that $D_0 + E$ and $D_1 + E$ are strongly effectively linearly equivalent. This can be summarized as follows: D_0 and D_1 on Γ are effectively linearly equivalent if and only if there exists a lifting of Γ to a smooth proper curve X/K of genus $h_1(\Gamma)$, and a finite morphism $\varphi : X \rightarrow \mathbf{P}^1$ such that $\tau_*(\varphi^{-1}(0)) = D_0 + E$ and $\tau_*(\varphi^{-1}(\infty)) = D_1 + E$ for some effective divisor E , where τ_* is the natural specialization map from $\text{Div}(X)$ to $\text{Div}(\Gamma)$.

Theorem 9.9. *The two notions of linear equivalence and effective linear equivalence of divisors on a metric graph Γ coincide. As a consequence, linear equivalence of divisors is the additive equivalence relation generated by (the retraction to Γ of) fibers of finite harmonic morphisms from a tropical modification of Γ to a metric graph of genus zero.*

Proof. Consider two divisors D_0 and D_1 which are effectively linearly equivalent. There exists an effective divisor E and a finite harmonic morphism $\varphi : \tilde{\Gamma} \rightarrow T$, from a tropical modification of Γ to a metric tree, such that $D_0 + E = D_{x_0}(\varphi)$ and $D_1 + E = D_{x_1}(\varphi)$ for two points $x_0, x_1 \in T$. By Proposition 9.8 we have $D_0 + E \sim D_1 + E$, which implies that D_0 and D_1 are linearly equivalent in $\tilde{\Gamma}$, and hence in Γ .

To prove the other direction, it will be enough to show that if D is linearly equivalent to zero, then there exists an effective divisor E such that $D + E$ and E are fibers of a finite harmonic morphism φ from a modification of Γ to a metric tree T , and such that φ can be lifted to a morphism $X \rightarrow \mathbf{P}^1$.

By assumption, there exists a rational function $f : \Gamma \rightarrow \mathbf{R} \cup \{\pm\infty\}$ such that $D + \operatorname{div}(f) = 0$. We claim that there is a tropical modification $\tilde{\Gamma}$ of Γ together with an extension of f to a (not necessarily finite) harmonic morphism $\varphi_0 : \tilde{\Gamma} \rightarrow \mathbf{R} \cup \{\pm\infty\}$. The tropical modification $\tilde{\Gamma}$ is obtained from Γ by choosing a vertex set which contains all the points in the support of D , adding an infinite edge to any finite vertex in Γ with $\operatorname{ord}_v(f) \neq 0$, and extending f as an affine linear function of slope $-\operatorname{ord}_v(f)$ along this infinite edge. It is clear that the resulting map φ_0 is harmonic.

Consider now the retraction map $\tau : \tilde{\Gamma} \rightarrow \Gamma$, and note that for the two divisors $D_{\pm\infty}(\varphi_0)$, we have $\tau_*(D_{\pm\infty}(\varphi_0)) = D_{\pm}$, where D_+ and D_- denote the positive and negative part of D , respectively. By Proposition 8.16, there exist tropical modifications $\bar{\Gamma}$ of $\tilde{\Gamma}$ and T of $\mathbf{R} \cup \{\pm\infty\}$ such that φ_0 extends to a finite harmonic morphism $\varphi : \bar{\Gamma} \rightarrow T$ which can be lifted to a finite morphism $X \rightarrow \mathbf{P}^1$. If we denote (again) the retraction map $\bar{\Gamma} \rightarrow \Gamma$ by τ , then $\tau_*(D_{\pm\infty}(\varphi)) = D_{\pm} + E_0$ for some effective divisor E_0 in Γ . Setting $E = D_- + E_0$, the divisors $D + E$ and E are strongly effectively linearly equivalent, and the theorem follows. \blacksquare

Example 9.10. Here is an example which illustrates the distinction between the notions of (effective) linear equivalence and strongly effective linear equivalence of divisors, as introduced above.

Let Γ be the metric graph depicted in Figure 13(a), with arbitrary lengths, and $K_{\Gamma} = (p) + (q)$ the canonical divisor on Γ .

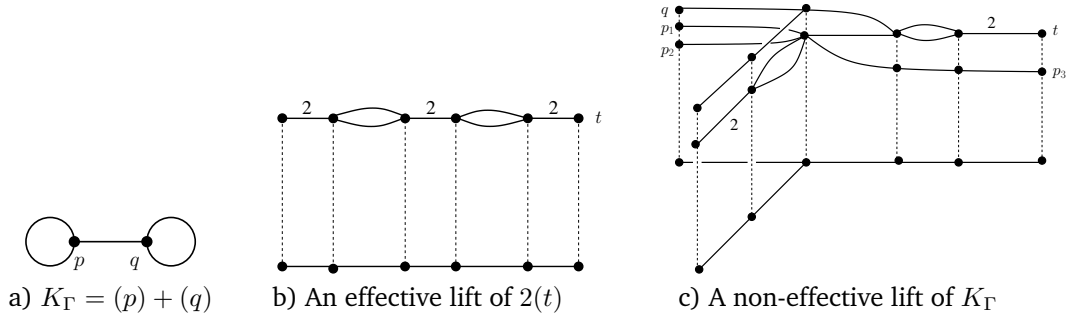


FIGURE 13.

We claim that K_{Γ} is not the specialization of any effective divisor of degree two representing the canonical class of a smooth proper curve of genus two over K . More precisely, we claim that for any triangulated punctured curve $(X, V \cup D)$ over K such that $\Sigma(X, V \cup D)$ is a tropical modification of Γ , and for any effective divisor \mathcal{D} in $\operatorname{Div}(X)$ with $K_{\Gamma} = \tau_*(\mathcal{D})$, we must have $r_X(\mathcal{D}) = 0$. (Here τ_* denotes the specialization map from $\operatorname{Div}(X)$ to $\operatorname{Div}(\Gamma)$ and $r_X(D) = \dim_K(H^0(X, \mathcal{O}(\mathcal{D}))) - 1$.) Indeed, otherwise there would exist a degree 2 finite harmonic morphism $\pi : \tilde{\Gamma} \rightarrow T$ from some tropical modification of Γ to a metric tree with the property that $\pi(p) = \pi(q)$. Restricting such a harmonic morphism to the preimage in $\tilde{\Gamma}$ of the loop containing p would imply, by Proposition 9.8, that the divisor (p) has rank one in a genus-one metric graph, which is impossible. On the other hand, Figure 13(b) shows that the divisor $2(t) \sim (p) + (q)$ can be lifted to an effective representative of the canonical class K_X , where t is the middle point of the loop edge with vertex q . This shows that the two linearly equivalent divisors $D_0 = (p) + (q)$ and $D_1 = 2(t)$ are not strongly effectively linearly equivalent.

However, D_0 and D_1 are effectively linearly equivalent. Indeed, adding $E = (p)$ to D_0 and D_1 , respectively, gives the two divisors $2(p) + (q)$ and $2(t) + (p)$ which are fibers of a degree 3 finite harmonic morphism from a tropical modification of Γ to a tree, as shown in Figure 13(c). Consequently, $D_0 + (p)$ and $D_1 + (p)$ can be lifted to linearly equivalent effective divisors on a smooth proper curve X .

Note also that Figure 13(b) shows that since $(p_1) + (p_2) + (q) - (p_3)$ can be lifted to a non-effective representative of the canonical class K_X , there exists a non-effective divisor \mathcal{D} in the canonical class K_X of X such that $\tau_*(\mathcal{D}) = (p) + (q)$.

9.11. Tame actions and quotients. Let \mathcal{C} be a metrized complex of k -curves, and denote by Γ the underlying metric graph of \mathcal{C} . An *automorphism* of \mathcal{C} is a (degree one) finite harmonic morphism of metrized complexes $h : \mathcal{C} \rightarrow \mathcal{C}$ which has an inverse. The group of automorphisms of \mathcal{C} is denoted by $\text{Aut}(\mathcal{C})$.

Let H be a finite subgroup of $\text{Aut}(\mathcal{C})$. The action of H on \mathcal{C} is *generically free* if for any vertex v of Γ , the inertia (stabilizer) group H_v acts freely on an open subset of C_v . A finite subgroup H of $\text{Aut}(\mathcal{C})$ is called *tame* if the action of H on \mathcal{C} is generically free and all the inertia subgroups H_x for x belonging to some C_v are cyclic of the form $\mathbb{Z}/d\mathbb{Z}$ for some positive integer d , with $(d, p) = 1$ if $p > 0$. In this case we say that the action of H on \mathcal{C} is tame.

In this section, we characterize tame group actions H on \mathcal{C} which lift to an action of H on some smooth proper curve X/K lifting \mathcal{C} . The main problem to consider is whether there exists a refinement $\tilde{\mathcal{C}}$ of \mathcal{C} and an extension of the action of H to $\tilde{\mathcal{C}}$ such that the quotient $\tilde{\mathcal{C}}/H$ can be defined, and such that the projection map $\pi : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}}/H$ is a tame harmonic morphism. The lifting of the action of H to a smooth proper curve X as above will then be a consequence of our lifting theorem.

Let H be a tame group of automorphisms of a metrized complex \mathcal{C} . Let W_H be the set of all $w \in \Gamma$ lying in the middle of an edge e such that there is an element $h \in H$ having w as an isolated fixed point. Denote by H_w the stabilizer of $w \in W_H$. It is easy to see that H_w consists of all elements h of H which restrict on e either to the identity or to the symmetry with center w . In particular, if $h|_e \neq \text{id}$, then h permutes the two vertices p and q adjacent to e . For $w \in W_H$, the inertia group $H_{\text{red}_p(e)} = H_{\text{red}_q(e)} \simeq \mathbb{Z}/d_e\mathbb{Z}$ (for some integer d_e) is a normal subgroup of index two in H_w :

$$0 \rightarrow H_{\text{red}_u(e)} \rightarrow H_w \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$

We make the following assumption on the groups H_w :

- (*) For all $w \in W_H$, the group H_w is isomorphic to the group generated by two elements σ and ζ with the relations

$$\sigma^2 = 1, \zeta^{d_e} = 1, \text{ and } \zeta\sigma\zeta = \sigma,$$

$$\text{and } H_{\text{red}_p(e)} = \langle \zeta \rangle.$$

The condition (*) means that the above short exact sequence splits, and the action of $\mathbb{Z}/2\mathbb{Z} \sim \{\pm 1\}$ on $H_{\text{red}_p(e)}$ is given by $h \rightarrow h^{\pm 1}$ for $h \in H_{\text{red}_p(e)}$.

We now claim that if Assumption (*) holds and $\text{char}(k) \neq 2$, then there exists a refinement $\tilde{\mathcal{C}}$ of \mathcal{C} such that the action of H on \mathcal{C} extends to a tame action of H on $\tilde{\mathcal{C}}$ with $W_H = \emptyset$.

To see this, fix an orientation of the edges of Γ , and for an oriented edge e , denote by p_0 and p_∞ the two vertices of Γ which form the tail and the head of e , respectively. Let w be a point lying in the middle of an oriented edge $e = (p_0, p_\infty)$ of Γ which is an isolated fixed point of some elements of H . Take the refinement $\tilde{\mathcal{C}}$ of \mathcal{C} obtained by adding all such points w to the vertex set of Γ and by setting $C_w = \mathbb{P}_k^1$, $\text{red}_e(\{w, p_0\}) = 0$, and $\text{red}_e(\{w, p_\infty\}) = \infty$. To see that the action of H on \mathcal{C} extends to $\tilde{\mathcal{C}}$, first note that one can define a generically free action of H_w on \mathbb{P}_k^1 (equivalently, one can embed H_w in $\text{Aut}(\mathbb{P}_k^1)$) in a way compatible with the action of H_w on Γ , i.e., such that all the elements of $H_{\text{red}_{p_0}(e)} = H_{\text{red}_{p_\infty}(e)}$ fix the two points 0 and ∞ of \mathbb{P}_k^1 , and such that the other elements of H_w permutes the two points $0, \infty \in \mathbb{P}_k^1$. Indeed, Assumption (*) is the necessary and sufficient condition for the existence of such an action. Under (*) and upon a choice of a $d_e = |H_{\text{red}_{p_0}(e)}|$ -th root of unity $\zeta_{d_e} \in k$, and upon the choice of the point $1 \in \mathbb{P}_k^1$ as a fixed point of σ , the actions of the two generators σ and ζ of H_w on \mathbb{P}_k^1 are given by $\sigma(z) = 1/z$ and $\zeta(z) = \zeta_{d_e}z$, respectively.

Fix once for all a d -th root of unity $\zeta_d \in k$ for each positive integer d (with $(d, p) = 1$ in the case $p > 0$). Given $h \in H$ and $w \in W_H$, we extend the action of H on \mathcal{C} to an action on $\tilde{\mathcal{C}}$ in the following

way: if $h(w) \neq w$, we define $h_w : C_w \rightarrow C_{h(w)}$ by $h = \text{id}_{\mathbf{P}^1}$; if $h \in H_w$, we define the action of h on C_w as above. This defines a generically free action of H on \tilde{C} . The inertia groups of the points $0, \infty$, and ± 1 in C_w are $\mathbf{Z}/d_e\mathbf{Z}$, $\mathbf{Z}/d_e\mathbf{Z}$, and $\mathbf{Z}/2\mathbf{Z}$, respectively. Since $p \neq 2$, this shows that the action of H on \tilde{C} is tame.

From now on, we assume that the action of H on C is tame and that no element of H has an isolated fixed point in the middle of an edge. We now define the quotient metrized complex C/H . The metric graph underlying C/H is the quotient graph Γ/H equipped with the following metric: given an edge e of Γ of length ℓ and stabilizer H_e , we define the length of its projection in Γ/H to be $\ell \cdot |H_e|$. The projection map $\Gamma \rightarrow \Gamma/H$ is a tame finite harmonic morphism.

For any vertex p of Γ , the k -curve associated to its image in C/H is C_p/H_p . The marked points of C_p/H_p are the different orbits of the marked points of C_p , and are naturally in bijection with the edges of Γ/H adjacent to the projection of p . The projection map $C \rightarrow C/H$ is a tame harmonic morphism of metrized complexes.

We can now formulate our main theorem on lifting tame group actions:

Theorem 9.12. *Let H be a finite group with a tame action on a metrized complex C .*

- (1) *If $W_H \neq \emptyset$, then Property $(*)$ and $\text{char}(k) \neq 2$ are the necessary and sufficient conditions for the existence of a refinement \tilde{C} of C such that the action of H on C extends to a tame action on \tilde{C} .*
- (2) *If $W_H = \emptyset$, then the quotient C/H exists in the category of metrized complexes. In addition, the action of H on C can be lifted to an action of H on a triangulated punctured K -curve $(X, V \cup D)$ such that $\Sigma(X, V \cup D_0) \simeq C$ with $D_0 \subset D$, the action of H on $X \setminus D$ is étale, and the inertia group H_x for $x \in D$ coincides with the inertia group $H_{\tau(x)}$ of the point $\tau(x) \in \Sigma(X, V \cup D_0) = C$.*

Proof. We already proved the first part. To see the second part, let C' be the (tropical) modification of C obtained as follows: for any closed point $x \in C_p$ with a non-trivial inertia group and which is not the reduction $\text{red}_p(e)$ of any edge e adjacent to p , consider the elementary tropical modification of C at x . Extend the action of H to a tame action on C' by defining $h_x : e_x \rightarrow e_{h(x)}$ to be affine with slope one for any such point. Let $\pi : C' \rightarrow C'/H$ be the projection map. Let $(X', V' \cup D')$ be a triangulated punctured K -curve such that $C(X', V' \cup D') \simeq C'/H$. By Theorem 7.4, the tame harmonic morphism π lifts to a morphism of triangulated punctured K -curves $(X, V \cup D) \rightarrow (X', V' \cup D')$. By Remark 7.5, we have an injection $\iota : \text{Aut}_{X'}(X) \hookrightarrow \text{Aut}_{C'/H}(C')$. By the construction given in the proof of Theorem 7.4, it is easy to see that every $h \in H$ lies in the image of ι , and thus $H \subset \text{Aut}_{X'}(X)$. The last part follows formally from the definition of the modification C' and the choice of X as the lifting of $\pi : C' \rightarrow C'/H$. \blacksquare

9.13. Characterization of liftable hyperelliptic augmented metric graphs. Let Γ be an augmented metric graph and denote by $r^\#$ the weighted rank function on divisors introduced in [AC]. Recall that this is the rank function on the non-augmented metric graph $\Gamma^\#$ obtained from Γ by attaching $g(p)$ cycles, called *virtual cycles*, of (arbitrary) positive lengths to each $p \in \Gamma$ with $g(p) > 0$. We say that an augmented metric graph Γ is *hyperelliptic* if $g(\Gamma) \geq 2$ and there exists a divisor D in Γ of degree two such that $r^\#_\Gamma(D) = 1$. An augmented metric graph is said to be *minimal* if it contains neither infinite vertices nor 1-valent vertices of genus 0. Every augmented metric graph Γ is tropically equivalent to a minimal augmented metric graph Γ' , which is furthermore unique if $g(\Gamma) \geq 2$. Since the tropical rank and weighted rank functions are invariant under tropical modifications, an augmented metric graph Γ is hyperelliptic if and only if Γ' is. Hence we restrict in this section to the case of minimal augmented metric graphs.

The following proposition is a refinement of a result from [Cha12] on vertex-weighted metric graphs (itself a strengthening of results from [BN09]):

Proposition 9.14. *For a minimal augmented metric graph Γ of genus at least two, the following assertions are equivalent:*

- (1) Γ is hyperelliptic;

- (2) *There exists an involution s on Γ such that:*
 - (a) *s fixes all the points $p \in \Gamma$ with $g(p) > 0$;*
 - (b) *the quotient Γ/s is a metric tree;*
- (3) *There exists an effective finite harmonic morphism of degree two $\varphi : \Gamma \rightarrow T$ from Γ to a metric tree T such that the local degree at any point $p \in \Gamma$ with $g(p) > 0$ is two.*

Furthermore if the involution s exists, then it is unique.

Proof. The implication (2) \Rightarrow (3) is obtained by taking $T = \Gamma/s$ and letting φ be the natural quotient map.

To prove (3) \Rightarrow (1), we observe that a finite harmonic morphism of degree two $\varphi : \Gamma \rightarrow T$ with local degree two at each vertex p with $g(p) > 0$ naturally extends to an effective finite harmonic morphism of degree two from a tropical modification Γ' of $\Gamma^\#$ to a tropical modification T' of T as follows: Γ' is obtained by modifying $\Gamma^\#$ once at the midpoint of each of its virtual cycles, and T' is obtained by modifying T precisely $g(p)$ times at each point $\varphi(p)$ with $g(p) > 0$. The map φ extends uniquely to an effective finite degree two harmonic morphism $\varphi' : \Gamma' \rightarrow T'$, since φ has local degree two at p whenever $g(p) > 0$. By Proposition 9.8, the linearly equivalent degree two divisors $D_x(\varphi')$ have rank one in Γ' as x varies over all points of T' , which shows that Γ is hyperelliptic.

It remains to prove (1) \Rightarrow (2). A *bridge edge* of Γ is an edge e such that $\Gamma \setminus e$ is not connected. Let Γ' be the augmented metric graph obtained by removing all bridge edges from Γ . Since Γ is minimal, any connected component of Γ' has positive genus. In particular the involution s , if exists, has to restrict to an involution on each such connected component. This implies that s has to fix pointwise any bridge edge. Hence we may now assume without loss of generality that Γ has no bridge edge. In this case s has the following simple definition: for any point $p \in \Gamma$, since $r_{\Gamma^\#}(D) = 1$ and Γ is two-edge connected, there exists a unique point $q = s(p)$ such that $D \sim (p) + (q)$. This also proves the uniqueness of the involution. \blacksquare

From now on we assume that $\text{char}(k) \neq 2$. An *involution* on a metrized complex \mathcal{C} is a finite harmonic morphism $s : \mathcal{C} \rightarrow \mathcal{C}$ with $s^2 = \text{id}_{\mathcal{C}}$. An involution is called *tame* if the action of the group generated by $\langle s \rangle \simeq \mathbf{Z}/2\mathbf{Z}$ on \mathcal{C} is tame.

If X/K is a (smooth proper) hyperelliptic curve, then the augmented metric graph Γ associated to stable model of X is hyperelliptic. Indeed if s_X is an involution on X , then the quotient map $X \rightarrow X/s$ tropicalizes to an effective tropical morphism $\varphi : \Gamma \rightarrow T$ of degree 2. The condition that φ has local degree 2 at each point p with $g(p) > 0$ comes from the fact that any non-constant algebraic map from a positive genus curve to \mathbf{P}^1 has degree at least two. The next theorem, combined with Proposition 9.14, provides a complete characterization of hyperelliptic augmented metric graphs which can be realized as the skeleton of a hyperelliptic curve over K .

Theorem 9.15. *Let Γ be a minimal hyperelliptic augmented metric graph, and let $s : \Gamma \rightarrow \Gamma$ be the involution given by Proposition 9.14 (2). Then the following assertions are equivalent:*

- (1) *There exists a hyperelliptic smooth proper curve X over K and an involution $s_X : X \rightarrow X$ such that Γ is the minimal skelton of X , and s coincides with the reduction of s_X to Γ .*
- (2) *For every $p \in \Gamma$ we have*

$$2g(p) \geq \kappa(p) - 2,$$

where $\kappa(p)$ denotes the number of tangent directions at p which are fixed by s .

Proof. Consider the finite harmonic morphism $\pi : \Gamma \rightarrow \Gamma/s$. We note that the tangent directions at p which are fixed by s are exactly those along which π has local degree two. Thus the condition $2g(p) \geq \kappa(p) - 2$ is equivalent to the ramification index R_p being non-negative, see Section 2. This proves (1) \Rightarrow (2).

To prove (2) \Rightarrow (1), we use Proposition 7.13 and Theorem 9.12. According to these results, it suffices to prove that the involution $s : \Gamma \rightarrow \Gamma$ lifts to an involution $\bar{s} : \mathcal{C} \rightarrow \mathcal{C}$ for some metrized complex \mathcal{C} with underlying augmented metric graph Γ such that \mathcal{C}/\bar{s} has genus zero. The existence of

such a lift follows from the observation that Hurwitz numbers of degree two are all positive (see (8.8), Remark 8.4). \blacksquare

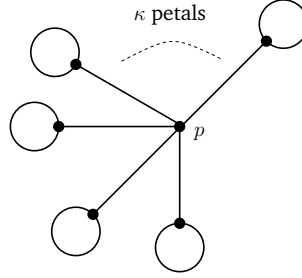


FIGURE 14.

Example 9.16. Let Γ be the augmented metric graph of genus g depicted in Figure 14 with arbitrary positive lengths. It is clearly hyperelliptic, and since the involution s restricts to the identity on each bridge edge, it fixes all tangent directions at p . Then one can lift Γ as a hyperelliptic curve of genus g if and only if $2g(p) \geq \kappa - 2$. In particular, if $g(p) = 0$ then this metric graph cannot be realized as the skeleton of a hyperelliptic curve as soon as $\kappa \geq 3$.

Since the hyperelliptic involution is unique for both curves and minimal augmented metric graphs, and since the tangent directions fixed by the hyperelliptic involution on an augmented metric graph correspond to bridge edges, we can reformulate Theorem 9.15 as follows, obtaining a metric strengthening of [Cap, Theorem 4.8]:

Corollary 9.17. *Let Γ be a minimal augmented metric graph of genus $g \geq 2$. Then there is a smooth proper hyperelliptic curve X over K of genus g having Γ as its minimal skeleton if and only if Γ is hyperelliptic and for every $p \in \Gamma$ the number of bridge edges adjacent to p is at most $2g(p) + 2$.*

10. GONALITY AND RANK

A fundamental (if vaguely formulated) question in tropical geometry is the following: If X is an algebraic variety and $\mathbf{T}X$ is a tropicalization of X (whatever it means), which properties of X can be read off from $\mathbf{T}X$? In this section, we discuss more precisely (for curves) the relation between the classical and tropical notions of gonality, and of the rank of a divisor. It is not difficult to prove that the gonality of a tropical curve (resp. the rank of a tropical divisor) provides a lower bound for the gonality (resp. an upper bound for the rank) of any lift (this is a consequence, for example, of Corollary 4.28). Here we address the question of sharpness for these inequalities:

- (1) Can a d -gonal (augmented or non-augmented) tropical curve C always be lifted to a d -gonal algebraic curve?
- (2) Can a divisor D on an (augmented or non-augmented) tropical curve C always be lifted to divisor of the same rank on an algebraic curve lifting C ?

It follows immediately from Theorem 8.9 that the answer to Question (1) is *yes* if C is not augmented, i.e., if we are allowed to arbitrarily increase the genus of finitely many points in C . On the other hand, we prove in this section that the answer to Question (1) in the case C is augmented, and the answer to Question (2) in both cases, is *no*.

We refer to [BN07, MZ08, AC, AB12] for the basic definitions concerning ranks of divisors on metric graphs, augmented metric graphs, and metrized complexes of curves.

10.1. Gonality of augmented graphs versus gonality of algebraic curves. An augmented tropical curve C is said to have an augmented (non-metric) graph G as its *combinatorial type* if C admits a representative whose underlying augmented graph is G . Given an augmented graph G , we denote

by $\mathcal{M}(G)$ the set of all augmented metric graphs which define a tropical curve C with combinatorial type G . In other words, $\mathcal{M}(G)$ consists of all augmented metric graphs which can be obtained by a finite sequence of tropical modifications (and their inverses) from an augmented metric graph Γ with underlying augmented graph G . When no confusion is possible, we identify an (augmented) tropical curve with any of its representatives as an (augmented) metric graph: in what follows, we deliberately write $C \in \mathcal{M}(G)$ for a tropical curve C with combinatorial type G . Note that the spaces $\mathcal{M}(G)$ appear naturally in the stratification of the moduli space of tropical curves of genus $g(G)$, see for example [Cap].

Definition 10.2. An augmented tropical curve C is called *d-gonal* if there exists a tropical morphism $C \rightarrow \mathbf{TP}^1$ of degree d .

An augmented graph G is called *stably d-gonal* if there exists a *d-gonal* augmented tropical curve C whose combinatorial type is G .

Remark 10.3. Our definition of the stable gonality of a graph is equivalent to the one given in [CKK], but is not the same as the notion of gonality defined in [Cap]. Here we are asking for the existence of an augmented metric graph $\Gamma \in \mathcal{M}(G)$ which admits an effective finite harmonic morphism of degree d to a metric tree.

In this section we prove the following theorem, which is an immediate consequence of Corollary 4.28 and Propositions 10.5 and 10.6 below.

Theorem 10.4. *There exists an augmented stably d-gonal graph G such that for any augmented metric graph $\Gamma \in \mathcal{M}(G)$ and any smooth proper connected K -curve X lifting Γ , the gonality of X is strictly larger than d .*

Let G_{27} be the graph depicted in Figure 15, which we promote to a totally degenerate augmented graph by taking the genus function to be identically zero. Note that $g(G_{27}) = 27$, and that $G_{27} \setminus \{p\}$ has three connected components, which we denote by A_1 , A_2 , and A_3 according to Figure 15.

Given an element $\Gamma \in \mathcal{M}(G_{27})$ and a tropical morphism $\varphi : C \rightarrow \mathbf{TP}^1$ from the tropical curve represented by Γ to \mathbf{TP}^1 , we denote by φ_i the restriction of φ to (the metric subgraph in Γ which corresponds to) A_i , and by φ_p the restriction of φ to a small neighborhood of the point p .

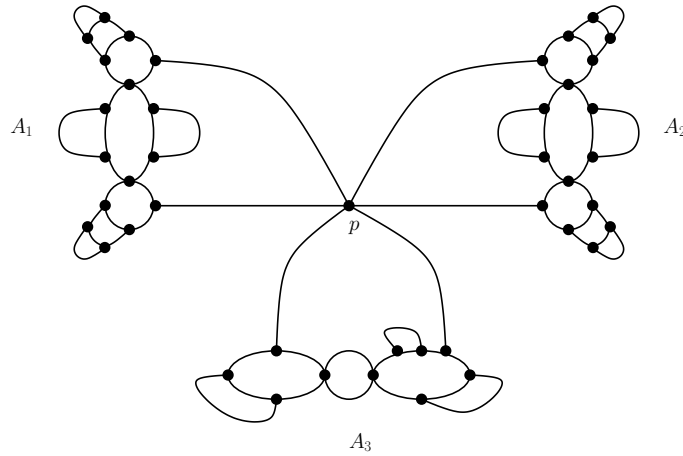


FIGURE 15. The graph G_{27}

Proposition 10.5. *The graph G_{27} depicted in Figure 15 is stably 4-gonal.*

Proof. We need to show the existence of a suitable tropical curve C with combinatorial type G_{27} which admits a tropical morphism of degree four to \mathbf{TP}^1 . For a suitable choice of edge lengths on G_{27} , we get an element $\Gamma \in \mathcal{M}(G_{27})$ such that there exists a harmonic morphism from Γ to a star-shaped

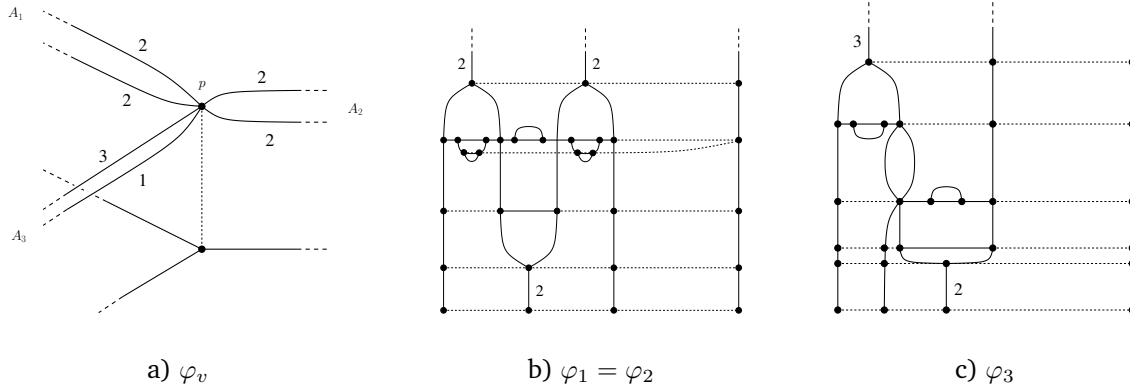
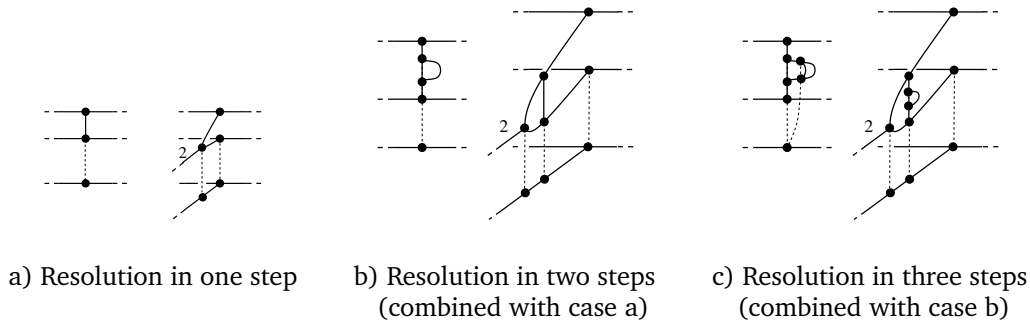


FIGURE 16. A tropical morphism of degree four.

genus zero augmented metric graph with three infinite edges, which has restrictions $\varphi_1, \varphi_2, \varphi_3, \varphi_v$ to A_1, A_2, A_3 , and a small neighborhood of p , respectively, given as in Figure 16. We claim that φ induces a tropical morphism, i.e., that there exists a tropical modification of φ which is finite and effective.

Note that each of the morphisms φ_1 and φ_2 contains a fiber of genus five, while the morphism φ_3 has two different fibers of genus one. All the other fibers of φ_1, φ_2 , and φ_3 are either finite or connected of genus zero. We depict in Figure 17 a few patterns which show how to resolve contractions of φ , turning φ into an augmented tropical morphism. Figure 17(a) shows how to resolve a contracted segment (resolving contracted fibers of genus zero). Figure 17(b) shows how to resolve a contracted cycle (resolving the contracted cycles in φ_3 and the middle contracted cycle in φ_1 and φ_2): the idea is to reduce to the case of a contracted segment, in which case one can use the resolution given in Figure 17(a) to finish. And finally, Figure 17(c) shows how to resolve the two contracted double-cycles in φ_1 and φ_2 by reducing to the case already treated in Figure 17(b). Note that performing these tropical modifications impose conditions on the length of the contracted edges in Γ , e.g., in Figure 17(b), the two edges adjacent to the contracted cycle should have the same length. Nevertheless, by appropriately choosing the edge lengths, we get the existence of a metric graph $\Gamma \in \mathcal{M}(G_{27})$ which admits a finite morphism of degree four to a metric tree. It is easily seen that this morphism is effective; thus we get a tropical curve C with combinatorial type G_{27} and a tropical morphism of degree four to \mathbf{TP}^1 , finishing the proof of the proposition. \blacksquare

FIGURE 17. Patterns to resolve contractions in the harmonic morphisms φ_1, φ_2 , and φ_3 .

To conclude the proof of Theorem 10.4, we now show the following:

Proposition 10.6. *There is no metrized complex of k -curves with underlying augmented metric graph in $\mathcal{M}(G_{27})$ and admitting a finite morphism of degree four to a metrized complex of k -curves of genus zero.*

We emphasize that the statement holds for any (algebraically closed) field k . The proof of Proposition 10.6 relies on some technical lemmas that we are now going to state.

We first recall a formula given in [AB12] for the rank of divisors on a metric graph $\Gamma = \Gamma_1 \vee \Gamma_2$ which is obtained as a wedge sum of two metric graphs Γ_1 and Γ_2 . Recall that given two metric graphs Γ_1 and Γ_2 and distinguished points $t_1 \in \Gamma_1$ and $t_2 \in \Gamma_2$, the *wedge sum* or *direct sum* of (Γ_i, t_i) , denoted $\Gamma = \Gamma_1 \vee \Gamma_2$, is the metric graph obtained by identifying the points t_1 and t_2 in the disjoint union of Γ_1 and Γ_2 . Denoting by $t \in \Gamma$ the image of t_1 and t_2 in Γ , one refers to $t \in \Gamma$ as a *cut-vertex* and to $\Gamma = \Gamma_1 \vee \Gamma_2$ as the decomposition corresponding to the cut-vertex t . (By abuse of notation, we will use t to denote both t_1 in Γ_1 and t_2 in Γ_2 .) There is an addition map $\text{Div}(\Gamma_1) \oplus \text{Div}(\Gamma_2) \rightarrow \text{Div}(\Gamma)$ which sends a pair of divisors D_1 and D_2 in $\text{Div}(\Gamma_1)$ and $\text{Div}(\Gamma_2)$ to the divisor $D_1 + D_2$ on Γ defined by pointwise addition of the coefficients in D_1 and D_2 .

Let $D_1 \in \text{Div}(\Gamma_1)$ and $D_2 \in \text{Div}(\Gamma_2)$. For any non-negative m , define $\eta_{\Gamma_1, D_1}(m)$ as minimum integer h such that $r_{\Gamma_1}(D_1 + h(t_1)) = m$. Then

$$(10.6.1) \quad r_{\Gamma}(D) = \min_{m \geq 0} \{m + r_{\Gamma_2}(D_2 - \eta_{\Gamma_1, D_1}(m)(t_2))\}.$$

(see [AB12] for details).

In what follows, equation (10.6.1) will be applied to a metric graph $\Gamma \in \mathcal{M}(A_1) = \mathcal{M}(A_2)$ (see Figure 18(a) and Lemma 10.7), to a metric graph $\Gamma \in \mathcal{M}(A_3)$ (see Figure 18(b) and Lemma 10.9), and to $\Gamma_{27} \in \mathcal{M}(G_{27})$ with cut-vertex p in the proof of Proposition 10.6.

Lemma 10.7. *Let Γ be a metric graph in $\mathcal{M}(A_1) = \mathcal{M}(A_2)$ as depicted in Figure 18(a). For any non-negative integers $a \leq 3$ and $b \leq 1$, the divisors $a(p) + b(q)$ and $b(p) + a(q)$ have rank zero in Γ .*

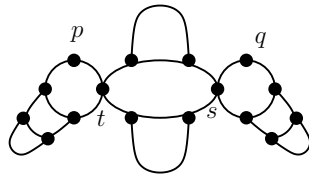
Proof. By symmetry it is enough to prove the lemma for the divisor $D = 3(p) + (q)$. Consider the decomposition $\Gamma = \Gamma_p \vee \Gamma_q$ associated to the cut-vertex t in Γ , where Γ_p and Γ_q denote the closure in Γ of the two connected components of $\Gamma \setminus \{t\}$ which contain the points p and q , respectively.

We claim that $\eta_{\Gamma_q, (q)}(1) = 3$. Assume for the moment that this is true. Then by (10.6.1), we have

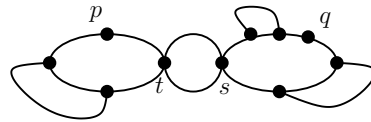
$$0 \leq r_{\Gamma}(3(p) + (q)) \leq 1 + r_{\Gamma_p}(3(p) - 3(t)).$$

By Lemma 10.8 below, in Γ_p we have $r_{\Gamma_p}(3(p) - 3(t)) = -1$. We thus infer that $r_{\Gamma}(3(p) + (q)) = 0$.

It remains to prove that $\eta_{\Gamma_q, (q)}(1) = 3$. In other words, we need to show that in Γ_q we have $r_{\Gamma_q}(2(t) + (q)) = 0$. For this, consider the decomposition $\Gamma_q = \Gamma_q^t \vee \Gamma_q^q$ corresponding to the cut-vertex s in Γ_q , where Γ_q^t and Γ_q^q denote the components which contain t and q , respectively. We claim that $\eta_{\Gamma_q^t, 2(t)}(1) = 1$. Assuming the claim, we have $0 \leq r_{\Gamma_q}(2(t) + (q)) \leq 1 + r_{\Gamma_q^q}((q) - (s)) = 0$ (since q and s are not linearly equivalent in Γ_q^q ; see Lemma 10.8). So it remains to prove that $\eta_{\Gamma_q^t, 2(t)}(1) = 1$. This is equivalent to $r_{\Gamma_q^t}(2(t)) = 0$, which is obviously the case. ■



a) A metric graph Γ in $\mathcal{M}(A_1) = \mathcal{M}(A_2)$



b) A metric graph Γ in $\mathcal{M}(A_3)$

FIGURE 18.

Lemma 10.8. *Let Γ be any metric graph in $\mathcal{M}(G_3)$, where G_3 is the totally degenerate graph depicted in Figure 19(a). Then the two divisors $3(p)$ and $3(t)$ are not linearly equivalent in Γ .*

Proof. By symmetry we can assume that the length of the edge $\{u, p\}$ is less than or equal to the length of the edge $\{t, w\}$. Then there exists a point t' in the segment $[t, w]$ so that $3(p) - 3(t) \sim 3(u) - 3(t')$ — see Figure 19(b) — and we are led to prove that $D = 3(u) - 3(t')$ is not linearly equivalent to zero. Consider the unique t' -reduced divisor $D_{t'}$ linearly equivalent to D in Γ (see e.g. [Ami, BN07] for the definition and basic properties of reduced divisors). It will be enough to show that $D_{t'} \neq 0$. Three cases can occur, depending on the lengths ℓ_z, ℓ_w , and $\ell_{t'}$ in Γ of the edges $\{u, z\}$, $\{u, w\}$, and the segment $\{u, t'\}$, respectively:

- If $\ell_z = \min\{\ell_z, \ell_u, \ell_{t'}\}$, then there are two points w' and t'' on the segments $\{u, w\}$ and $\{u, t'\}$, respectively, such that $D_{t'} = (z) + (w') + (t'') - 3(t')$.
- If $\ell_u = \min\{\ell_z, \ell_u, \ell_{t'}\}$, then there are two points z' and t'' on the segments $\{u, z\}$ and $\{u, t'\}$, respectively, such that $D_{t'} = (z') + (w) + (t'') - 3(t')$.
- If $\ell_{t'} = \min\{\ell_z, \ell_u, \ell_{t'}\}$, then there are two points z' and w' on the segments $\{u, z\}$ and $\{u, w\}$, respectively, such that $D_{t'} = (z') + (w') - 2(t')$.

In all the three cases, we have $D_{t'} \neq 0$, which shows that D cannot be equivalent to zero in Γ . ■

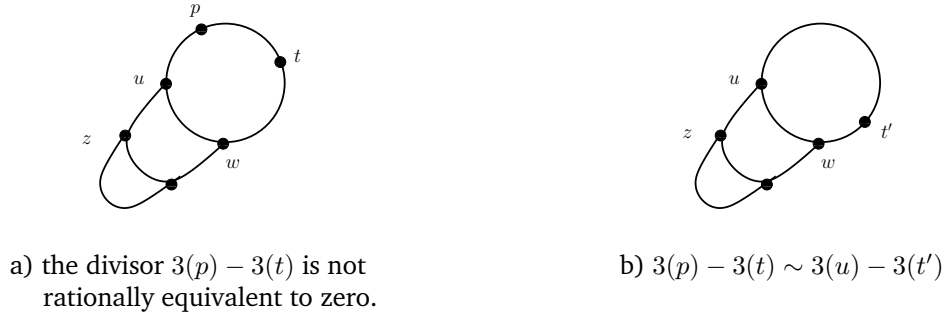


FIGURE 19.

Lemma 10.9. *Let $\Gamma \in \mathcal{M}(A_3)$ be a metric graph as depicted in Figure 18(b). For any $a, b \leq 2$, the divisor $a(p) + b(q)$ has rank zero on Γ .*

Proof. The arguments are similar to the ones used in the proof of Lemma 10.7. Consider the cut-vertex t in Γ and denote by Γ_p and Γ_q the corresponding components containing p and q , respectively. We claim that $\eta_{\Gamma_q, 2(q)}(1) = 2$. This obviously implies the lemma. Indeed, $r_{\Gamma_p}(2(p) - 2(t)) = -1$ (which can be verified by an analogue of Lemma 10.8 in Γ_p), and thus (10.6.1) implies that $r_\Gamma(2(p) + 2(q)) \leq 1 + r_{\Gamma_p}(2(p) - 2(t)) = 0$.

To show that $\eta_{\Gamma_q, 2(q)}(1) = 2$, it will be enough to show that $r_{\Gamma_q}(2(q) + (t)) = 0$. This can be done in exactly the same way by considering the other cut-vertex s adjacent to t in Γ_q . ■

Lemma 10.10. *Let x_1, x_2 , and x_3 be distinct points in $\mathbf{P}^1(k)$. Then there does not exist a morphism $f : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ of degree four branched over x_1, x_2 and x_3 and having ramification profile $(2, 2)$, $(2, 2)$, and $(3, 1)$ at these three points.*

Proof. Suppose that such a rational map $f : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ exists. The monodromy group of f is a subgroup of \mathfrak{S}_4 , so its cardinality is of the form $2^a 3^b$. In particular, if the characteristic of k is neither 2 nor 3, then f has a tame monodromy group and the non-existence of f then comes from the fact that $H_{0,0}^4((2, 2), (2, 2), (3, 1)) = 0$ (see Example 8.4).

Hence it remains to check the lemma for $\text{char}(k) = 2, 3$. Note that the same technique we use in this case works in any characteristic, but the computations are a bit more tedious in characteristic different from 2 and 3.

Up to the action of $\mathrm{GL}(2, k)$ on \mathbf{P}^1 via automorphisms, we may assume that $x_1 = 0$, $x_2 = \infty$, and $x_3 = 1$, and that

$$f(X) = a \frac{X^2(X+1)^2}{(X+b)^2}$$

with $a \neq 0$ and $b \neq 0, -1, \infty$. Hence the condition on the ramification profile of x_3 translates as

$$aX^2(X+1)^2 - (X+b)^2 = c(X-d)^3(X-e)$$

with $c \neq 0$, $d \neq 0, -1, \infty, b$, and $e \neq 0, -1, \infty, b, d$. Looking at the coefficients of the two polynomials, we obtain the following five equations

$$(E_1) : a = c, \quad (E_2) : e = -2 - 3d, \quad (E_4) : a - 1 = 3cd(d + e),$$

$$(E_4) : -2b = cd^2(d + 3e), \quad (E_5) : -b^2 = cd^3e.$$

If k has characteristic 2, then (E_2) becomes $e = d$ which contradicts the fact that $e \neq d$.

If k has characteristic 3, then these five equations become

$$(E_1) : a = c, \quad (E_2) : e = -2 = 1, \quad (E_4) : a = 1, \quad (E_4) : b = d^3, \quad (E_5) : b^2 = d^3,$$

and so (E_4) together with (E_5) gives $b = 1 = e$, in contradiction with our assumptions. \blacksquare

We can now give the promised proof of Proposition 10.6.

Proof. (Proof of Proposition 10.6) Suppose that there exists a metrized complex of k -curves \mathcal{C}_{27} of genus 27 with underlying augmented metric graph Γ_{27} in $\mathcal{M}(G_{27})$, and admitting a finite harmonic morphism of metrized complexes of degree four $\varphi : \mathcal{C}_{27} \rightarrow \mathcal{T}$, for \mathcal{T} of genus zero with underlying metric tree denoted by T . Without loss of generality, we may assume that T has no infinite vertex $q \in V_\infty(T)$ such that any infinite edge e' adjacent to an infinite vertex $q' \in \varphi^{-1}(q)$ has $d_{e'}(\varphi) = 1$.

We are going to prove below that the local degree at p is 4. Assuming that this is the case, we show how the proposition follows. Denote by Γ_1, Γ_2 , and Γ_3 the three components of $\Gamma_{27} \setminus \{p\}$ which contain A_1, A_2 , and A_3 , respectively. Since the degree of φ at p is four, we have $\varphi^{-1}(\varphi(p)) = \{p\}$. Therefore, by the connectivity of Γ_i , the images of Γ_i under φ are pairwise disjoint in T . This shows that for x sufficiently close to $\varphi(p)$ in T , the support of the divisor $D_x(\varphi)$ lives entirely in one of the Γ_i for $i \in \{1, 2, 3\}$. Choose x_i sufficiently close to $\varphi(p)$ such that the support of $D_{x_i}(\varphi)$ is contained in Γ_i . Applying Proposition 9.8, we see that each divisor $D_{x_i}(\varphi)$ has rank one in Γ_i . Now, according to Lemma 10.7, the degree-four divisor $D_{x_1}(\varphi)$ (resp. $D_{x_2}(\varphi)$) must be of the form $2(a) + 2(b)$ for two points a and b sufficiently close to p and lying on the two different branches of Γ_1 (resp. Γ_2) adjacent to p . Similarly, by Lemma 10.9, the divisor $D_{x_3}(\varphi)$ has to be of the form $3(a) + (b)$ for two points a and b sufficiently close to p and lying on the two different branches of Γ_3 adjacent to p . This shows that the map φ_p , the restriction of φ to a sufficiently small neighborhood of p in Γ_{27} , coincides with the map depicted in Figure 10.5(a). The proposition now follows from Lemma 10.10.

It remains to prove that $d_p(\varphi) = 4$. We first claim that φ maps one of the components Γ_i , for $i = 1, 2, 3$, onto a connected component of $T \setminus \{\varphi(p)\}$. Otherwise, for the sake of contradiction, suppose that $\varphi^{-1}(\varphi(p))$ consists of p and one point p_i in each of the components Γ_i for $i = 1, 2, 3$. Then φ has local degree one at each of the points p_i . By Proposition 9.8, $D_{\varphi(p)}(\varphi) = (p) + (p_1) + (p_2) + (p_3)$ has rank one in Γ . By equation (10.6.1) applied to the cut-vertex p in Γ_{27} , we infer that the divisor $(p) + (p_i)$ has rank one in the metric graph $\bar{\Gamma}_i$, the closure of Γ_i in Γ_{27} . In other words, the metric graphs $\bar{\Gamma}_i$ are hyperelliptic, which is clearly not the case. This gives a contradiction and the claim follows.

Summarizing, there must exist at least one Γ_i such that φ maps Γ_i onto one of the connected components of $T \setminus \{\varphi(p)\}$. Reasoning again as in the first part of the proof, it follows from Proposition 9.8 and Lemmas 10.7 and 10.9 that the restriction of φ to Γ_i has degree four, which implies that $d_p(\varphi) = 4$. \blacksquare

10.11. Lifting divisors of given rank. First, recall that to a smooth proper curve X over K together with a semistable vertex set V and a subset D_0 of $X(K)$ compatible with V , we can naturally associate a metrized complex of curves $\mathcal{C} = \Sigma(X, V \cup D_0)$ with underlying augmented metric graph Γ . As in [AB12], there are natural specialization maps on divisors, which we denote for simplicity by the same letter τ_* :

$$\tau_* : \text{Div}(X) \rightarrow \text{Div}(\mathcal{C}), \quad \text{and} \quad \tau_* : \text{Div}(\mathcal{C}) \rightarrow \text{Div}(\Gamma).$$

Since this discussion is pointless in the case of rational curves, we may assume that X (equivalently, \mathcal{C} or the augmented metric graph Γ) has positive genus. We will also assume that Γ does not have any infinite vertices, i.e., that D_0 is empty, which does not lead to any real loss of generality and which makes various statements easier to write down and understand. We may also assume without loss of generality that V is a strongly semistable vertex set of X .

According to the Specialization Inequality [Bak08, AC, AB12]), for any divisor D in $\text{Div}(X)$ one has

$$(10.11.1) \quad r_X(D) \leq r_{\mathcal{C}}(\tau_*(D)) \leq r_{\Gamma}^{\#}(\tau_*(D)) \leq r_{\Gamma}(\tau_*(D)),$$

where r_X , $r_{\mathcal{C}}$ and r_{Γ} denote rank of divisors on X , \mathcal{C} and (unaugmented) Γ , respectively, and $r_{\Gamma}^{\#}$ denotes the weighted rank in the augmented metric graph (Γ, g) (see (9.13)).

We spend the rest of this section discussing the sharpness of the inequalities appearing in (10.11.1).

Definition 10.12. Let \mathcal{C} be a metrized complex of curves whose underlying metric graph Γ has no infinite leaves, and let \mathcal{D} be a Λ -rational divisor in $\text{Div}_{\Lambda}(\mathcal{C})$. A *lifting* of the pair $(\mathcal{C}, \mathcal{D})$ consists of a triple $(X, V; D_X)$ where X is a smooth proper curve over K , V is a strongly semistable vertex set for which $\mathcal{C} = \Sigma(X, V)$, and D_X is a divisor in $\text{Div}(X)$ with $\mathcal{D} \sim \tau_*(D_X)$. We say that the inequality $r_X \leq r_{\mathcal{C}}$ is *sharp* if for any metrized complex of curves \mathcal{C} and any divisor $\mathcal{D} \in \text{Div}(\mathcal{C})$, there exists a lifting $(X, V; D_X)$ of $(\mathcal{C}, \mathcal{D})$ such that $r_X(D_X) = r_{\mathcal{C}}(\mathcal{D})$.

We can define in a similar way what it means to lift a divisor on an (augmented) metric graph to a divisor on a metrized complex of curves or to a smooth proper curve over K , and what it means for the corresponding specialization inequalities to be sharp.

It is easy to see that the inequality $r_{\Gamma}^{\#} \leq r_{\Gamma}$ is not sharp (see [AB12] for a precise formula relating the two rank functions). The following example is due to Ye Luo (unpublished); we thank him for his permission to include it here. Together with Corollary 4.28, it implies that the inequality $r_X \leq r_{\Gamma}$ is not sharp.

Example 10.13. (Luo) Let Γ be a metric graph in $\mathcal{M}(G_7)$, where G_7 is the graph of genus seven depicted in Figure 20(a), such that all edge lengths in Γ are equal, and let $D = (p) + (q) + (s) \in \text{Div}(\Gamma)$. Then $r_{\Gamma}(D) = 1$, however there does not exist any finite harmonic morphism of metric graphs $\varphi : \Gamma \rightarrow T$ of degree three to a metric tree such that $D_x(\varphi)$ (the fiber of φ above x) is linearly equivalent to D for some $x \in T$.

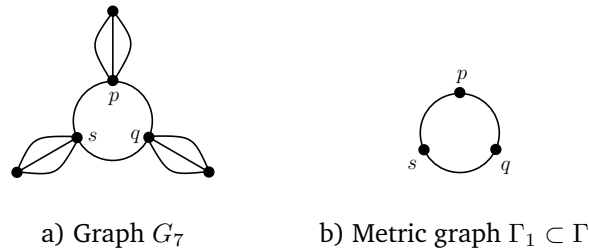


FIGURE 20.

We briefly sketch a proof. Suppose that such a finite harmonic morphism $\varphi : \Gamma \rightarrow T$ exists for some $\Gamma \in \mathcal{M}(G_7)$. Since Γ is not hyperelliptic, one easily verifies that $D_{\varphi(p)}(\varphi) = 3(p)$, $D_{\varphi(q)}(\varphi) = 3(q)$,

and $D_{\varphi(s)}(\varphi) = 3(s)$. This shows the existence of a finite morphism $\varphi' : \Gamma_1 \rightarrow T'$ of degree 3 to metric tree T' where Γ_1 is the metric graph depicted in Figure 20(b) with the three edges having the same length, so that $D_{\varphi'(p)}(\varphi') = 3(p)$, $D_{\varphi'(q)}(\varphi') = 3(q)$, and $D_{\varphi(s)}(\varphi') = 3(s)$. But it is easy to verify by hand that such a morphism φ' does not exist.

Proposition 10.14. *Neither of the inequalities $r_X \leq r_C$ and $r_C \leq r_\Gamma^\#$ is sharp.*

Proof. To show the non-sharpness of the inequality $r_X \leq r_C$, let \mathcal{C} be a metrized complex of curves whose underlying metric graph Γ belongs to the family depicted in Figure 14, with first Betti number κ , and whose genus function is positive at each vertex. Consider the divisor $\mathcal{D}_d = d(p) \oplus d(x)$ in \mathcal{C} for a closed point x in C_p and d a positive integer. If d is sufficiently large compared to the genera of the vertices, then $r_C(\mathcal{D}_d) \geq 1$. If the pair $(\mathcal{C}, \mathcal{D}_d)$ lifted to a triple $(X, V; D_X)$ with $\tau_*(D_X) \sim \mathcal{D}_d$, then there would exist a finite harmonic morphism $\varphi : \tilde{\mathcal{C}} \rightarrow \mathcal{T}$ from a modification of \mathcal{C} to a metrized complex of curves of genus zero. But this would imply the existence of a degree d morphism $\varphi_p : C_p \rightarrow \mathbf{P}^1$ such that the image of red_p (on edges adjacent to p in Γ) is contained in the set of critical values of φ_p . By the Riemann-Hurwitz formula, this is impossible for κ large enough compared to d .

To show the non-sharpness of the inequality $r_C \leq r_\Gamma^\#$, let again (Γ, g) be an augmented metric graph with underlying graph depicted in Figure 14 with $\kappa \geq 3$ and $2 \leq 2g(p) < \kappa - 2$, and let $D = 2(p)$. One easily computes that $r_\Gamma^\#(D) = 1$. An algebraic curve of genus $g(p) \geq 1$ contains at most $2g(p) + 2$ distinct points p such that $2(p)$ is in a given linear system of degree two, which implies that (Γ, g) cannot be lifted to a hyperelliptic metrized complex of curves. This shows that the inequality $r_C \leq r_\Gamma^\#$ is not sharp. ■

REFERENCES

- [AB12] O. Amini and M. Baker, *Linear series on metrized complexes of algebraic curves*, Preprint available at <http://arxiv.org/abs/1204.3508>.
- [AC] O. Amini and L. Caporaso, *Riemann-Roch theory for weighted graphs and tropical curves*, to appear in Adv. Math., Preprint available at <http://arxiv.org/abs/1112.5134>.
- [ACP12] S. Abramovich, L. Caporaso, and S. Payne, *The tropicalization of the moduli space of curves*, Preprint available at <http://arxiv.org/abs/1212.0373>.
- [Ami] O. Amini, *Reduced divisors and embeddings of tropical curves*, to appear in Trans. Amer. Math. Soc., Preprint available at <http://arxiv.org/abs/1007.5364>.
- [Ana00] C.K. Anand, *A discrete analogue of the harmonic morphism*, Harmonic morphisms, harmonic maps, and related topics (Brest, 1997), Chapman & Hall/CRC Res. Notes Math., vol. 413, Chapman & Hall/CRC, Boca Raton, FL, 2000, pp. 109–112.
- [Bak08] M. Baker, *Specialization of linear systems from curves to graphs*, Algebra Number Theory **2** (2008), no. 6, 613–653, With an appendix by Brian Conrad.
- [BBM11] B. Bertrand, E. Brugallé, and G. Mikhalkin, *Tropical open Hurwitz numbers*, Rend. Semin. Mat. Univ. Padova **125** (2011), 157–171.
- [BdlHN97] R. Bacher, P. de la Harpe, and T. Nagnibeda, *The lattice of integral flows and the lattice of integral cuts on a finite graph*, Bull. Soc. Math. France **125** (1997), no. 2, 167–198.
- [Ber90] V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990.
- [Ber93] ———, *Étale cohomology for non-Archimedean analytic spaces*, Inst. Hautes Études Sci. Publ. Math. **78** (1993), 5–161.
- [Ber99] ———, *Smooth p -adic analytic spaces are locally contractible*, Invent. Math. **137** (1999), no. 1, 1–84.
- [BF11] M. Baker and X. Faber, *Metric properties of the tropical Abel-Jacobi map*, J. Algebraic Combin. **33** (2011), no. 3, 349–381.
- [BGR84] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 261, Springer-Verlag, Berlin, 1984.
- [BL85] S. Bosch and W. Lütkebohmert, *Stable reduction and uniformization of abelian varieties. I*, Math. Ann. **270** (1985), no. 3, 349–379.
- [BL93] ———, *Formal and rigid geometry. I. Rigid spaces*, Math. Ann. **295** (1993), no. 2, 291–317.
- [BM] E. Brugallé and G. Mikhalkin, *Realizability of superabundant curves*, In preparation.
- [BN07] M. Baker and S. Norine, *Riemann-Roch and Abel-Jacobi theory on a finite graph*, Adv. Math. **215** (2007), no. 1, 766–788.

- [BN09] ———, *Harmonic morphisms and hyperelliptic graphs*, International Math. Research Notices **15** (2009), 2914–2955.
- [BPR11] M. Baker, S. Payne, and J. Rabinoff, *Non-Archimedean geometry, tropicalization, and metrics on curves*, 2011, Preprint available at <http://arxiv.org/abs/1104.0320>.
- [BR10] M. Baker and R. Rumely, *Potential theory and dynamics on the Berkovich projective line*, Mathematical Surveys and Monographs, vol. 159, American Mathematical Society, Providence, RI, 2010.
- [BR13] M. Baker and J. Rabinoff, *An analytic version of Raynaud’s description of component groups of Néron models*, in preparation (2013).
- [Cap] L. Caporaso, *Gonality of algebraic curves and graphs*, Preprint available at <http://arxiv.org/abs/1201.6246>.
- [Cha12] M. Chan, *Tropical hyperelliptic curves*, J. Alg. Combin. **15** (2012), 2914–2955.
- [CJM10] R. Cavalieri, Paul Johnson, P., and H. Markwig, *Tropical Hurwitz numbers*, J. Algebraic Combin. **32** (2010), no. 2, 241–265.
- [CKK] G. Cornelissen, F. Kato, and J. Kool, *A combinatorial Li-Yau inequality and rational points on curves*, Preprint available at <http://arxiv.org/abs/1211.2681>.
- [CLD12] A. Chambert-Loir and A. Ducros, *Formes différentielles sur les espaces de Berkovich*, Preprint available at <http://arxiv.org/abs/1204.6277>.
- [Col03] R. F. Coleman, *Stable maps of curves*, Documenta Math. **Extra volume Kato** (2003), 217–225.
- [Con99] B. Conrad, *Irreducible components of rigid spaces*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 2, 473–541.
- [Duc08] A. Ducros, *Triangulations et cohomologie étale sur une courbe analytique p -adique*, J. Algebraic Geom. **17** (2008), no. 3, 503–575.
- [EKS84] A. Edmonds, R. Kulkarni, and R. Stong, *Realizability of branched coverings of surfaces*, Trans. Amer. Math. Soc. **282** (1984), no. 2, 773–790.
- [Fab11a] X. Faber, *Topology and geometry of the Berkovich ramification locus for rational functions*, 2011, Preprint available at <http://arxiv.org/abs/1102.1432>.
- [Fab11b] ———, *Topology and geometry of the Berkovich ramification locus for rational functions, II*, 2011, Preprint available at <http://arxiv.org/abs/1104.0943>.
- [HM82] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), no. 1, 23–88, With an appendix by W. Fulton.
- [Hus62] D. Husemoller, *Ramified coverings of Riemann surfaces*, Duke Math. J. **29** (1962), 167–174.
- [KS01] M. Kontsevich and Y. Soibelman, *Homological mirror symmetry and torus fibrations*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publishing, River Edge, NJ, 2001, pp. 203–263.
- [KS06] ———, *Affine structures and non-Archimedean analytic spaces*, The unity of mathematics, Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 321–385.
- [Liu06] Q. Liu, *Stable reduction of finite covers of curves*, Compositio Math. **142** (2006), no. 1, 101–118.
- [LL99] Q. Liu and D. Lorenzini, *Models of curves and finite covers*, Compositio Math. **118** (1999), no. 1, 61–102.
- [LZ04] S. Lando and A. Zvonkin, *Graphs on surfaces and their applications*, Encyclopaedia of Mathematical Sciences, Low-Dimensional Topology, II., vol. 141, Springer-Verlag, 2004, With an appendix by Don B. Zagier.
- [Mik] G. Mikhalkin, *Phase-tropical curves I. Realizability and enumeration*, In preparation.
- [Mik05] ———, *Enumerative tropical algebraic geometry in \mathbb{R}^2* , J. Amer. Math. Soc. **18** (2005), no. 2, 313–377.
- [Mik06] ———, *Tropical geometry and its applications*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 827–852.
- [MZ08] G. Mikhalkin and I. Zharkov, *Tropical curves, their Jacobians and theta functions*, Curves and abelian varieties, Contemp. Math., vol. 465, Amer. Math. Soc., Providence, RI, 2008, pp. 203–230.
- [Pay09] S. Payne, *Analytification is the limit of all tropicalizations*, Math. Res. Lett. **16** (2009), no. 3, 543–556.
- [PP06] E. Pervova and C. Petronio, *On the existence of branched coverings between surfaces with prescribed branch data. I*, Algebr. Geom. Topol. **6** (2006), 1957–1985 (electronic).
- [PP08] ———, *On the existence of branched coverings between surfaces with prescribed branch data. II*, J. Knot Theory Ramifications **17** (2008), no. 7, 787–816.
- [Ray70] M. Raynaud, *Spécialisation du foncteur de Picard*, Inst. Hautes Études Sci. Publ. Math. (1970), no. 38, 27–76.
- [Sai96] M. Saidi, *Revêtements étales abéliens, courants sur les graphes et réductions semi-stable des courbes*, Manuscripta Mathematica **89** (1996), 245–265.
- [Sai97] ———, *Revêtements modérés et groupe fondamental de graphe de groupes*, Compositio Mathematica **107** (1997), 319–338.
- [SGA1] A. Grothendieck and M. Raynaud, *Revêtements étales et groupe fondamental (SGA 1)*, Séminaire de Géométrie Algébrique de l’Institut des Hautes Études Scientifiques (SGA 1), Société Mathématique de France, Paris, 2003.
- [Tem] M. Temkin, *Introduction to Berkovich analytic spaces*, Preprint available at <http://arxiv.org/abs/1010.2235v1>.
- [Thu05] A. Thuillier, *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d’Arakelov*, Ph.D. thesis, University of Rennes, 2005, Preprint available at <http://tel.ccsd.cnrs.fr/documents/archives0/00/01/09/90/index.html>.
- [Ura00] H. Urakawa, *A discrete analogue of the harmonic morphism and green kernel comparison theorems*, Glasg. Math. J. **42** (2000), 319–334.

- [Wel13] J. Welliaveetil, *A Riemann-Hurwitz formula for skeleta in non-Archimedean geometry*, 2013, Preprint available at <http://arxiv.org/abs/1303.0164>.
- [Wew99] S. Wewers, *Deformation of tame admissible covers of curves*, Aspects of Galois theory (Gainesville, FL, 1996), London Math. Soc. Lecture Note Ser., vol. 256, Cambridge Univ. Press, Cambridge, 1999, pp. 239–282.

E-mail address: oamini@math.ens.fr

CNRS-DMA, ÉCOLE NORMALE SUPÉRIEURE, 45 RUE D'ULM, PARIS

E-mail address: mbaker@math.gatech.edu

SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA GA 30332-0160, USA

E-mail address: brugalle@math.jussieu.fr

UNIVERSITÉ PIERRE ET MARIE CURIE, PARIS 6, 4 PLACE JUSSIEU, 75 005 PARIS, FRANCE

E-mail address: rabinoﬀ@math.harvard.edu

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA 02138, USA